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APPROXIMATION OF ABSTRACT DIFFERENTIAL EQUATIONS

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1. Introduction

This review paper is devoted to the numerical analysis of abstract differential equations in Banach spaces. Most of the finite difference, finite element, and projection methods can be considered from the point of view of general approximation schemes (see, e.g., [207,210,211] for such a representation). Results obtained for general approximation schemes make the formulation of concrete numerical methods easier and give an overview of methods which are suitable for different classes of problems.

The qualitative theory of differential equations in Banach spaces is presented in many brilliant papers and books. We can refer to the bibliography [218], which contains about 3000 references. Unfortunately, no books or reviews on general approximation theory appear for differential equations in abstract spaces during last 20 years. Any information on the subject can be found in the original papers only. It seems that such a review is the first step towards describing a complete picture of discretization methods for abstract differential equations in Banach spaces.

In Sec. 2 we describe the general approximation scheme, different types of convergence of operators, and the relation between the convergence and the approximation of spectra. Also, such a convergence analysis can be used if one considers elliptic problems, i.e., the problems which do not depend on time.

Section 3 contains a complete picture of the theory of discretization of semigroups on Banach spaces. It summarizes Trotter–Kato and Lax–Richtmyer theorems from the general and common point of view and related problems.

The approximation of ill-posed problems is considered in Sec. 4, which is based on the theory of approximation of local C-semigroups. Since the backward Cauchy problem is very important in applications and admits a stochastic noise, we also consider approximation using a stochastic regularization. Such an approach was never considered in the literature before to the best of our knowledge.

In Sec. 5, we present discrete coercive inequalities for abstract parabolic equations in $C_{\tau_n}([0,T];E_n)$, $C^{\alpha}_{\tau_n}([0,T];E_n)$, $L^p_{\tau_n}([0,T];E_n)$, and $B_{\tau_n}([0,T];C^{\alpha}(\Omega_h))$ spaces.

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The last section, Sec. 6 deals with semilinear problems. We consider approximations of Cauchy problems and also the problems with periodic solutions. The approach described here is based on the theory of rotation of vector fields and the principle of compact approximation of operators.

2. General Approximation Scheme

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators on E will be denoted by $\mathcal{C}(E)$. We denote by $\sigma(B)$ the spectrum of the operator B, by $\rho(B)$ the resolvent set of B, by $\mathcal{N}(B)$ the null space of B, and by $\mathcal{R}(B)$ the range of B. Recall that $B \in B(E)$ is called a Fredholm operator if $\mathcal{R}(B)$ is closed, $\dim \mathcal{N}(B) < \infty$ and $\operatorname{codim} \mathcal{R}(B) < \infty$, the index of B is defined as $\operatorname{ind} B = \dim \mathcal{N}(B) - \operatorname{codim} \mathcal{R}(B)$. The general approximation scheme [83–85, 187, 207, 210] can be described in the following way. Let E_n and E be Banach spaces, and let $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E, E_n)$, $n \in \mathbb{N} = \{1, 2, \cdots\}$, with the following property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E.$$

Definition 2.1. A sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-*convergent* to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$; we write this as $x_n \to x$.

Definition 2.2. A sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-*compact* if for any $\mathbb{N}' \subseteq \mathbb{N}$ there exist $\mathbb{N}'' \subseteq \mathbb{N}'$ and $x \in E$ such that $x_n \to x$, as $n \to \infty$ in $\mathbb{N}''$.

Definition 2.3. A sequence of bounded linear operators $B_n \in B(E_n), n \in \mathbb{N}$, is said to be $\mathcal{PP}$-*convergent* to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, such that $x_n \to x$ one has $B_n x_n \to Bx$. We write this as $B_n \to B$.

For general examples of notions of $\mathcal{P}$-convergence, see [82, 187, 203, 211].

Remark 2.1. If we set $E_n = E$ and $p_n = I$ for each $n \in \mathbb{N}$, where I is the identity operator on E, then Definition 2.1 leads to the traditional pointwise convergence of bounded linear operators which is denoted by $B_n \to B$.

Denote by E^+ the positive cone in a Banach lattice E. An operator B is said to be *positive* if for any $x^+ \in E^+$, it follows $Bx^+ \in E^+$; we write $0 \preceq B$.

Definition 2.4. A system $\{p_n\}$ is said to be *discrete order preserving* if for all sequences $\{x_n\}, x_n \in E_n$, and any element $x \in E$, the following implication holds: $x_n \to x$ implies $x_n^+ \to x^+$.

It is known [99] that $\{p_n\}$ preserves the order iff $\|p_n x^+ - (p_n x)^+\|_{E_n} \to 0$ as $n \to \infty$ for any $x \in E$.

In the case of unbounded operators, and, in general, we know infinitesimal generators are unbounded, we consider the notion of *compatibility.*

Definition 2.5. A sequence of closed linear operators $\{A_n\}$, $A_n \in \mathcal{C}(E_n)$, $n \in \mathbb{N}$, is said to be *compatible* with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in \mathbb{N}$, such that $x_n \to x$ and $A_n x_n \to Ax$. We write this as (A_n, A) are compatible.

In practice, Banach spaces E_n are usually finite dimensional, although, in general, say, for the case of a closed operator A, we have $\dim E_n \to \infty$ and $\|A_n\|_{B(E_n)} \to \infty$ as $n \to \infty$.

2.1. Approximation of spectrum of linear operators. The most important role in approximations of equation $Bx = y$ and approximations of spectra of an operator B is played by the notions of *stable* and *regular* convergence. These notions are used in different areas of numerical analysis (see [10, 15, 81, 86–89, 207, 210, 212, 222]).

Definition 2.6. A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in \mathbb{N}$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \to B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1), n \to \infty$. We will write this as: $B_n \to B$ stably.

Definition 2.7. A sequence of operators $\{B_n\}, B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ iff $B_n \to B$ and the following implication holds:

$$\|x_n\|_{E_n} = O(1) \ \& \ \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact} \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact}.$$

We write this as: $B_n \to B$ regularly.

Theorem 2.1 ([210])**.** *For $B_n \in B(E_n)$ and $B \in B(E)$ the following conditions are equivalent:*

(i) $B_n \to B$ *regularly,* B_n *are Fredholm operators of index* 0 *and* $\mathcal{N}(B) = \{0\}$;

(ii) $B_n \to B$ *stably and* $\mathcal{R}(B) = E$;

(iii) $B_n \to B$ *stably and regularly;*

(iv) *if one of conditions* (i)–(iii) *holds, then there exist* $B_n^{-1} \in B(E_n), B^{-1} \in B(E)$, *and* $B_n^{-1} \to B^{-1}$ *regularly and stably.*

This theorem admits an extension to the case of closed operators $B \in \mathcal{C}(E), B_n \in \mathcal{C}(E_n)$ [213].

Let $\Lambda \subseteq \mathbb{C}$ be some open connected set, and let $B \in B(E)$. For an isolated point $\lambda \in \sigma(B)$, the corresponding maximal invariant space (or generalized eigenspace) will be denoted by $\mathcal{W}(\lambda; B) = P(\lambda)E$, where $P(\lambda) = \dfrac{1}{2\pi i} \int_{|\zeta-\lambda|=\delta} (\zeta I - B)^{-1} d\zeta$ and δ is small enough so that there are no points of $\sigma(B)$ in the disc $\{\zeta : |\zeta - \lambda| \le \delta\}$ different from λ. The isolated point $\lambda \in \sigma(B)$ is a *Riesz point* of B if

$\lambda I - B$ is a Fredholm operator of index zero and $P(\lambda)$ is of finite rank. Denote by $\mathcal{W}(\lambda, \delta; B_n) = \bigcup\limits_{|\lambda_n - \lambda| < \delta, \lambda_n \in \sigma(B_n)} \mathcal{W}(\lambda_n, B_n)$, where $\lambda_n \in \sigma(B_n)$ are taken from a δ-neighborhood of λ. It is clear that $\mathcal{W}(\lambda, \delta; B_n) = P_n(\lambda) E_n$, where $P_n(\lambda) = \dfrac{1}{2\pi i} \int\limits_{|\zeta - \lambda| = \delta} (\zeta I_n - B_n)^{-1} d\zeta$. The following theorems state the complete picture of the approximation of the spectrum.

Theorem 2.2 ([82, 208, 209])**.** *Assume that $L_n(\lambda) = \lambda I - B_n$ and $L(\lambda) = \lambda I - B$ are Fredholm operators of index zero for any $\lambda \in \Lambda$ and $L_n(\lambda) \to L(\lambda)$ stably for any $\lambda \in \rho(B) \cap \Lambda \neq \emptyset$. Then*

(i) *for any $\lambda_0 \in \sigma(B) \cap \Lambda$, there exists a sequence $\{\lambda_n\}$, $\lambda_n \in \sigma(B_n)$, $n \in \mathbb{N}$, such that $\lambda_n \to \lambda_0$ as $n \to \infty$;*

(ii) *if for some sequence $\{\lambda_n\}, \lambda_n \in \sigma(B_n)$, $n \in \mathbb{N}$, one has $\lambda_n \to \lambda_0 \in \Lambda$ as $n \to \infty$, then $\lambda_0 \in \sigma(B)$;*

(iii) *for any $x \in \mathcal{W}(\lambda_0, B)$, there exists a sequence $\{x_n\}$, $x_n \in \mathcal{W}(\lambda_0, \delta; B_n)$, $n \in \mathbb{N}$, such that $x_n \to x$ as $n \to \infty$;*

(iv) *there exists $n_0 \in \mathbb{N}$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) \geq \dim \mathcal{W}(\lambda_0, B)$ for any $n \geq n_0$.*

Remark 2.2. The inequality in (iv) can be strict for all $n \in \mathbb{N}$ as is shown in [207].

Theorem 2.3 ([210])**.** *Assume that $L_n(\lambda)$ and $L(\lambda)$ are Fredholm operators of index zero for all $\lambda \in \Lambda$. Suppose that $L_n(\lambda) \to L(\lambda)$ regularly for any $\lambda \in \Lambda$ and $\rho(B) \cap \Lambda \neq \emptyset$. Then statements* (i)–(iii) *of Theorem* 2.2 *hold and also*

(iv) *there exists $n_0 \in \mathbb{N}$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) = \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$;*

(v) *any sequence $\{x_n\}, x_n \in \mathcal{W}(\lambda_0, \delta; B_n)$, $n \in \mathbb{N}$, with $\|x_n\|_{E_n} = 1$ is $\mathcal{P}$-compact and any limit point of this sequence belongs to $\mathcal{W}(\lambda_0, B)$.*

Remark 2.3. Estimates of $|\lambda_n - \lambda_0|$, gap $\Big(\mathcal{W}(\lambda_0, \delta; B_n), \mathcal{W}(\lambda_0, B)\Big)$ and $|\hat{\lambda}_n - \lambda_0|$ are given in [210], where $\hat{\lambda}_n$ denotes the arithmetic mean (counting algebraic multiplicities) of the spectral values of B_n that contribute to $\mathcal{W}(\lambda_0, \delta; B_n)$. For the notion of *gap* and its properties, see [105].

2.2. Regions of convergence. Theorems 2.2 and 2.3 have been generalized to the case of closed operators in [213] by using the following notions introduced by Kato [105].

Definition 2.8. The region of stability $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \rho(A_n)$ for almost all n and such that the sequence $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n \in \mathbb{N}}$ is bounded. The region of convergence $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \Delta_s(\{A_n\})$ and such that the sequence of operators $\{(\lambda I_n - A_n)^{-1}\}_{n \in \mathbb{N}}$ is $\mathcal{PP}$-convergent to some operator $S(\lambda) \in B(E)$.

It is clear that $S(\cdot)$ is a pseudo-resolvent, and $S(\cdot)$ is a resolvent of some operator iff $\mathcal{N}(S(\lambda)) = \{0\}$ for some λ (cf. [105]).

Definition 2.9. A sequence of operators $\{K_n\}$, $K_n \in \mathcal{C}(E_n)$, is called regularly compatible with an operator $K \in \mathcal{C}(E)$ if (K_n, K) are compatible and, for any bounded sequence $\|x_n\|_{E_n} = O(1)$ such that $x_n \in D(K_n)$ and $\{K_n x_n\}$ is $\mathcal{P}$-compact, it follows that $\{x_n\}$ is $\mathcal{P}$-compact, and the $\mathcal{P}$-convergence of $\{x_n\}$ to some x and that of $\{K_n x_n\}$ to some y as $n \to \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$ imply that $x \in D(K)$ and $Kx = y$.

Definition 2.10. The region of regularity $\Delta_r = \Delta_r(\{A_n\}, A)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that (K_n, K), where $K_n = \lambda I_n - A_n$ and $K = \lambda I - A$ are regularly compatible.

The relationships between these regions are given by the following statement.

Proposition 2.1 ([213]). *Suppose that $\Delta_c \neq \emptyset$ and $\mathcal{N}(S(\lambda)) = \{0\}$ at least for one point $\lambda \in \Delta_c$ so that $S(\lambda) = (\lambda I - A)^{-1}$. Then (A_n, A) are compatible and*

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r = \Delta_r \cap \rho(A).$$

It is shown in [213] that the conditions (A_n, A) *are compatible,* $\lambda I - A_n$ *and* $\lambda I - A$ *are Fredholm operators with index zero for any* $\lambda \in \Lambda$ *and* $\rho(A) \cap \Lambda \neq \emptyset$ imply (i)–(iv) of Theorem 2.2 when $\rho(A) \cap \Lambda \subseteq \Delta_s$ and imply (i)–(iii) of Theorem 2.2 and (iv)–(v) of Theorem 2.3, when $\Lambda \subseteq \Delta_r$.

Definition 2.11. A Riesz point $\lambda_0 \in \sigma(A)$ is said to be strongly stable in *Kato's* sense if $\dim \mathcal{W}(\lambda_0, \delta; B_n) \leq \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$.

Theorem 2.4 ([213]). *The Riesz point $\lambda_0 \in \sigma(A)$ is strongly stable in Kato's sense iff $\lambda_0 \in \Lambda \cap \Delta_r \cap \sigma(A)$.*

Investigations of approximation of spectra and types of convergence, but not those of general approximation scheme are given in [13, 40, 52, 130, 131, 142, 145].

2.3. Convergence in Anselone's conditions. Throughout this subsection we assume that $E_n = E$ and $p_n = I$ for all $n \in \mathbb{N}$. Hence the symbol $\mathcal{P}$ will be omitted in the notation of this subsection.

Let us recall that if $B_n \to B$ compactly (see Definition 2.12), then for any $\lambda \neq 0$ we have $\lambda I - B_n \to \lambda I - B$ regularly [207]. When $B_n \to B$ compactly and B is a compact operator, Anselone [10] has proved that

$$\|(B_n - B)B_n\| \to 0, \ \|(B_n - B)B\| \to 0 \ \text{ as } n \to \infty. \tag{2.1}$$

Considering an approximation of a weakly singular compact integral operator, Ahues [4] has proved that these convergence properties (2.1) are sufficient to state that a Riesz point is strongly stable in Kato's sense.

Theorem 2.5 ([7]). *Assume that $B \in B(E)$ is compact and that $B_n \to B$. If $\|(B_n - B)B_n\| \to 0$ as $n \to \infty$; then for any nonzero $\lambda_0 \in \sigma(B)$, assertions* (i)–(iii) *of Theorem* 2.2 *and assertions* (iv)–(v) *of Theorem* 2.3 *hold.*

Theorem 2.6 ([7]). *Assume that $B_n \to B$ and* (2.1) *holds. Then for any nonzero Riesz point $\lambda_0 \in \sigma(B)$, assertions* (i)–(iii) *of Theorem* 2.2 *and assertions* (iv)–(v) *of Theorem* 2.3 *hold.*

Corollary 2.1 ([5]). *Assume that $B_n \to B$, $\lambda I - B_n$ are Fredholm operators of index zero for $\lambda \in \{z : |z - \lambda_0| \le \delta\}$, and $\|(B_n - B)B_n^k\| \to 0$ as $n \to \infty$ for some $k \in \mathbb{N}$. Then for any nonzero Riesz point $\lambda_0 \in \sigma(B)$, assertions* (i)–(iii) *of Theorem* 2.2 *and assertions* (iv)–(v) *of Theorem* 2.3 *hold.*

Theorem 2.7 ([7]). *Assume that B is compact, $B_n \to B$, and $\|B_n(B_n - B)\| \to 0$. Then $\lambda_0 I_n - B_n \to \lambda_0 I - B$ regularly for any $\lambda_0 \neq 0$.*

Theorem 2.8 ([7]). *Assume that B is compact, $B_n \to B$, and $\|B_n^k(B_n - B)\| \to 0$ for some $k \in \mathbb{N}$. Then $\lambda_0 I_n - B_n \to \lambda_0 I - B$ regularly for any $\lambda_0 \neq 0$.*

Let $r(B)$ be a spectral radius of operator $B \in B(E)$.

Theorem 2.9 ([18]). *Let E be a Banach lattice. Let $0 \preceq B_n, B \in B(E)$ be such that $B_n \to B$ and $\|(B_n - B)^+\| \to 0$ as $n \to \infty$. Suppose that $r(B)$ is a Riesz point of $\sigma(B)$. Then $r(B_n)$ is a Riesz point of $\sigma(B_n)$ and $r(B_n) \to r(B)$ as $n \to \infty$.*

The conclusion on the order of convergence of eigenvectors in Theorem 2.9 also is given in [17].

The application of Theorems 2.5–2.8 to the numerical solution of a mathematical model used in the jet printer industry is considered in [6, 118].

2.4. Compact convergence of resolvents. We now consider the important class of operators which have compact resolvents. We will use this property of generator as an assumption in Sec. 6. In this case, it is natural to consider approximate operators which “preserve” this property.

Definition 2.12. A sequence of operators $\{B_n\}$, $B_n \in B(E_n)$, $n \in \mathbb{N}$, *converges compactly* to an operator $B \in B(E)$ if $B_n \to B$ and the following compactness condition holds:

$$\|x_n\|_{E_n} = O(1) \Longrightarrow \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

Definition 2.13. *The region of compact convergence of resolvents,* $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly.

Theorem 2.10. *Assume that $\Delta_{cc} \neq \emptyset$. Then for any $\zeta \in \Delta_s$ the following implication holds:*

$$\|x_n\|_{E_n} = O(1) \ \&\ \|(\zeta I_n - A_n)x_n\|_{E_n} = O(1) \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact.} \tag{2.2}$$

Conversely, if for some $\zeta \in \Delta_c \cap \rho(A)$ implication (2.2) holds, then $\Delta_{cc} \neq \emptyset$.

Proof. Let $(\mu I_n - A_n)^{-1} \to (\mu I - A)^{-1}$ compactly for some $\mu \in \Delta_{cc}$. Then for $\|x_n\|_{E_n} = O(1)$ and $\|(\zeta I - A_n)x_n\|_{E_n} = O(1)$, from the Hilbert identity

$$(\zeta I_n - A_n)^{-1} - (\mu I_n - A_n)^{-1} = (\mu - \zeta)(\zeta I_n - A_n)^{-1}(\mu I_n - A_n)^{-1}, \tag{2.3}$$

we obtain $x_n = (\mu I_n - A_n)^{-1}(\zeta I_n - A_n)x_n - (\zeta - \mu)(\mu I_n - A_n)^{-1}x_n$, and it follows that $\{x_n\}$ is $\mathcal{P}$-compact. Conversely, let implication (2.2) hold for some $\zeta_0 \in \Delta_c \cap \rho(A)$. We show that $\zeta_0 \in \Delta_{cc}$. Taking a bounded sequence $\{y_n\}$, $n \in \mathbb{N}$, we obtain $\|(\zeta_0 I_n - A_n)^{-1}y_n\|_{E_n} = O(1)$ for $n \in \mathbb{N}$. Let us apply implication (2.2) to the sequence $x_n = (\zeta_0 I_n - A_n)^{-1}y_n$. It is easy to see that $\{x_n\}$ is $\mathcal{P}$-compact. Hence $\zeta_0 \in \Delta_{cc}$. ☐

Corollary 2.2. *Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_{cc} = \Delta_c \cap \rho(A)$.*

Proof. It is clear that $\Delta_{cc} \subseteq \Delta_c \cap \rho(A)$. To prove that $\Delta_{cc} \supseteq \Delta_c \cap \rho(A)$, let us consider the Hilbert identity (2.3). Now let $\mu \in \Delta_{cc}$. Then $\mu \in \Delta_{cc} \cap \Delta_c \cap \rho(A)$. Hence, for every $\zeta \in \Delta_c \cap \rho(A)$ and for any bounded sequence $\{x_n\}$, $n \in \mathbb{N}$, the sequence $\{(\zeta I_n - A_n)^{-1}x_n\}$ is $\mathcal{P}$-compact. ☐

Comparing Definitions 2.7, 2.8, and 2.13 with implication (2.2), we see that $\Delta_{cc} \subseteq \Delta_r$.

Theorem 2.11. *Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_r = \mathbb{C}$.*

Proof. Take any point $\lambda_1 \in \mathbb{C}$. We have to show that $(\lambda_1 I_n - A_n, \lambda_1 I - A)$ are regularly compatible. Assume that $\|x_n\|_{E_n} = O(1)$ and that $\{(\lambda_1 I_n - A_n)x_n\}$ is $\mathcal{P}$-compact. To show that $\{x_n\}$ is $\mathcal{P}$-compact, we take $\mu \in \Delta_{cc}$. Using (2.3) with $\zeta = \lambda_1$, we obtain $x_n = (\mu I_n - A_n)^{-1}(\lambda_1 I_n - A_n)x_n + (\lambda_1 - \mu)(\mu I_n - A_n)^{-1}x_n$ and, therefore, $\{x_n\}$ is $\mathcal{P}$-compact. Assume now that $x_n \to x$ and $(\lambda_1 I_n - A_n)^{-1}x_n \to y$, as $n \to \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$. Then $x = (\mu I - A)^{-1}y - (\lambda_1 - \mu)(\mu I - A)^{-1}x$, and it follows that $x \in D(A)$ and $(\lambda_1 I - A)x = y$. ☐

3. Discretization of Semigroups

Let us consider the following well-posed Cauchy problem in the Banach space E with an operator $A \in \mathcal{C}(E)$

$$\begin{aligned} &u'(t) = Au(t),\ t \in [0, \infty), \\ &u(0) = u^0, \end{aligned} \tag{3.1}$$

where the operator A generates a C_0-semigroup $\exp(\cdot A)$. It is well-known that this C_0-semigroup gives the solution of (3.1) by the formula $u(t) = \exp(tA)u^0$ for $t \geq 0$. The theory of well-posed problems and numerical analysis of these problems have been developed extensively; see, e.g., [75, 88, 105, 161, 163, 200, 216].

Let us consider on the general discretization scheme for the semidiscrete approximation of the problem (3.1) in the Banach spaces E_n:

$$\begin{aligned} &u_n'(t) = A_n u_n(t), \ t \in [0, \infty), \\ &u_n(0) = u_n^0, \end{aligned} \tag{3.2}$$

with the operators $A_n \in \mathcal{C}(E_n)$ such that they generate C_0-semigroups which are compatible with the operator A and $u_n^0 \to u^0$.

3.1. The simplest discretization schemes. We have the following version of Trotter–Kato's Theorem on the general approximation scheme.

Theorem 3.1 ([203] (Theorem ABC))**.** *The following conditions (A) and (B) are equivalent to condition* (C).

(A) *Compatibility. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge*:

$$(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1};$$

(B) *Stability. There are some constants $M \geq 1$ and ω, independent of n and that $\|\exp(tA_n)\| \leq M\exp(\omega t)$ for $t \geq 0$ and any $n \in \mathbb{N}$;*

(C) *Convergence. For any finite $T > 0$, one has $\max_{t\in[0,T]} \|\exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$ whenever $u_n^0 \to u^0$.*

The analytic C_0-semigroup case is slightly different from the general case but has the same property (A).

Theorem 3.2 ([161])**.** *Let operators A and A_n generate analytic C_0-semigroups. The following conditions* (A) *and (B_1) are equivalent to condition (C_1).*

(A) *Compatibility. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge*:

$$(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1};$$

(B_1) *Stability. There are some constants $M_2 \geq 1$ and ω_2 such that*

$$\|(\lambda I - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|}, \quad \operatorname{Re}\lambda > \omega_2, n \in \mathbb{N};$$

(C_1) *Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$, we have*

$$\max_{\eta \in \Sigma(\theta,\mu)} \| \exp(\eta A_n)u_n^0 - p_n \exp(\eta A)u^0 \| \to 0$$

as $n \to \infty$ whenever $u_n^0 \to u^0$. Here, $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$ and $\Sigma(\theta) = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$.

Definition 3.1. A linear operator $A : D(A) \subseteq E \to E$ is said to have the *positive off-diagonal* (POD) *property* if $\langle Au, \phi \rangle \geq 0$ whenever $0 \preceq u \in D(A)$ and $0 \preceq \phi \in E^*$ with $\langle u, \phi \rangle = 0$.

Definition 3.2. An element $e \in E^+$ is said to be an order-one in E if for every $x \in E$ there exists $0 \leq \lambda \in \mathbb{R}$ such that $-\lambda e \preceq x \preceq \lambda e$. For $e \in \operatorname{int} E^+$ we can define the order-one norm by

$$\|x\|_e = \inf\{\lambda \geq 0 : -\lambda e \preceq x \preceq \lambda e\}.$$

An ordered Banach space E is called an order-one space if there exists $e \in \operatorname{int} E^+$ such that $\| \cdot \|_E = \| \cdot \|_e$.

Now we can state a version of the Trotter–Kato theorem for positive semigroups.

Theorem 3.3 ([169]). *Let the operators A_n and A from* (3.1) *and (3.2) be compatible, let E, E_n be order-one spaces, and let $e_n \in D(A_n) \cap \operatorname{int} E_n^+$. Assume that the operators A_n have the* POD *property and $A_n e_n \preceq 0$ for sufficiently large n. Then $\exp(tA_n) \to \exp(tA)$ uniformly in $t \in [0, T]$.*

We can assume without loss of generality that conditions (A) and (B) hold for the corresponding semigroup case if any discretization processes in time are considered. If we denote by $T_n(\cdot)$ a family of discrete semigroups as in [105], i.e., $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ and $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = \left[\frac{t}{\tau_n}\right]$, as $\tau_n \to 0, n \to \infty$, then one obtains the following assertion.

Theorem 3.4 ([203] (Theorem ABC-discr.)). *The following conditions* (A) *and* (B′) *are equivalent to condition* (C′).

(A) *Compatibility. There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(\breve{A}_n)$ such that the resolvents converge:*

$$(\lambda I_n - \breve{A}_n)^{-1} \to (\lambda I - A)^{-1};$$

(B′) *Stability. There are some constants $M_1 \geq 1$ and ω_1 such that*

$$\|T_n(t)\| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{\mathbb{R}}_+ = [0, \infty), n \in \mathbb{N};$$

(C′) *Convergence. For any finite $T > 0$ one has $\max_{t \in [0,T]} \|T_n(t)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \to u^0$.*

Theorem 3.5 ([203]). *Assume that conditions* (A) *and* (B) *of Theorem 3.1 hold. Then the implicit difference scheme*

$$\frac{\overline{U}_n(t+\tau_n)-\overline{U}_n(t)}{\tau_n}=A_n\overline{U}_n(t+\tau),\, \overline{U}_n(0)=u_n^0, \tag{3.3}$$

is stable, i.e. $\|(I_n-\tau_nA_n)^{-k_n}\|\le M_1e^{\omega_1t}, t=k_n\tau_n\in\overline{\mathbb{R}}_+$, *and gives an approximation of the solution of problem* (3.1), *i.e.,* $\overline{U}_n(t)\equiv(I_n-\tau_nA_n)^{-k_n}u_n^0\to\exp(tA)u_n^0$ $\mathcal{P}$*-converges uniformly with respect to* $t=k_n\tau_n\in[0,T]$ *as* $u_n^0\to u^0, n\to\infty, k_n\to\infty, \tau_n\to 0$.

Here, in Theorem 3.5, $\breve{A}_n=A_n(I_n-\tau_nA_n)^{-1}$, and, therefore, $(I_n-\tau_nA_n)^{-k_n}=(I_n+\tau_n\breve{A}_n)^{k_n}$.

Theorem 3.6 ([203]). *Assume that conditions* (A) *and* (B) *of the Theorem* 3.1 *hold and condition*

$$\tau_n\|A_n^2\|=O(1) \tag{3.4}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t+\tau_n)-U_n(t)}{\tau_n}=A_nU_n(t),\;\; U_n(0)=u_n^0, \tag{3.5}$$

is stable, i.e., $\|(I_n+\tau_nA_n)^{k_n}\|\le Me^{\omega t}, t=k_n\tau_n\in\overline{\mathbb{R}}_+$, *and gives an approximation of the solution of problem* (3.1), *i.e.,* $U_n(t)\equiv(I_n+\tau_nA_n)^{k_n}u_n^0\to u(t)$ $\mathcal{P}$*-converges uniformly with respect to* $t=k_n\tau_n\in[0,T]$ *as* $n\to\infty, k_n\to\infty, \tau_n\to 0$.

Theorem 3.7 ([161]). *Assume that conditions* (A) *and* (B_1) *of Theorem* 3.2 *hold and condition*

$$\tau_n\|A_n\|\le 1/(M+2), n\in\mathbb{N} \tag{3.6}$$

is fulfilled. Then the difference scheme (3.5) *is stable and gives an approximation of the solution of problem* (3.1), *i.e.,* $U_n(t)\equiv(I_n+\tau_nA_n)^{k_n}u_n^0\to u(t)$ *discretely* $\mathcal{P}$*-converge uniformly with respect to* $t=k_n\tau_n\in[0,T]$ *as* $u_n^0\to u^0,\ n\to\infty,\ k_n\to\infty,\ \tau_n\to 0$.

Let us introduce the following conditions:

(B_1') *Stability.* There are constants M' and ω' such that

$$\|\exp(tA_n)\|\le M'e^{\omega' t},\ \|A_n\exp(tA_n)\|\le\frac{M'}{t}e^{\omega' t},\ t\in\mathbb{R}_+.$$

(B_1'') *Stability.* There are constants M'',ω'', and $\tau^*>0$ such that

$$\|(I_n-\tau_nA_n)^{-k}\|\le M''e^{\omega'' k\tau_n},\ \|k\tau_nA_n(I_n-\tau_nA_n)^{-k}\|\le M''e^{\omega'' k\tau_n},\ 0<\tau_n<\tau^*, n,k\in\mathbb{N}.$$

Proposition 3.1 ([183]). *Conditions* (B_1), (B_1'), *and* (B_1'') *are equivalent.*

Theorem 3.8. *Conditions* (A) *and* (B_1'') *are equivalent to condition* (C_1).

Theorem 3.9 ([164]). *Let the assumptions of Theorem 3.7 and (3.4) be satisfied. Then*

$$tA_n(I_n + \tau_n A_n)^{k_n} \to tA\exp(tA) \text{ uniformly in } t = k_n\tau_n \in [0, T]. \tag{3.7}$$

Conversely, if $(I_n+\tau_n A_n)^{k_n} \to \exp(tA)$ *uniformly in* $t = k_n\tau_n \in [0,T]$ *and* (3.7) *is satisfied, then condition* (C_1) *holds.*

Theorem 3.10 ([164]). *Let condition* (B_1) *hold. Then*

$$\|\exp(tA_n) - (I_n - \tau_n A_n)^{-k_n}\| \leq c\frac{\tau_n}{t}e^{\omega t}.$$

If, moreover, the stability condition (3.6) *holds, then*

$$\begin{aligned}
\|\exp(tA_n) - (I_n + \tau_n A_n)^{k_n}\| &\leq c\frac{\tau_n}{t}e^{\omega t}, \\
\|(\exp(tA_n) - (I_n + \tau_n A_n)^{k_n})x_n\| &\leq c\tau_n e^{\omega t}\|A_n x_n\|, \\
\|A_n(\exp(tA_n) - (I_n + \tau_n A_n)^{k_n})x_n\| &\leq c\frac{\tau_n}{t}e^{\omega t}\|A_n x_n\|, \qquad t = k_n\tau_n.
\end{aligned}$$

In the case of analytic C_0-semigroups for the forward scheme, as we saw, the stability condition

$$\tau_n\|A_n\| < 1/(M+2)$$

cannot be improved even in Hilbert spaces for self-adjoint operators. In the case of almost periodic C_0-semigroups and the forward scheme for differential equations of first order in time (3.1), one obtains necessary and sufficient stability condition $\tau_n\|A_n\| < 1$ [163]. It was discovered that the stability condition of the forward scheme like (3.5) for the positive C_0-semigroups also can be written in the form $\tau_n\|A_n\| < 1$; see [168]. Stability of difference schemes under some spectral conditions were obtained in [26].

The stability of difference schemes for differential equations in Hilbert spaces in the energy norm are investigated in [179, 180], where schemes with weights were also considered. Semidiscrete approximations are studied also in [180].

3.2. Rational approximation. Let us denote by $P_p(z)$ an element of the set of all real polynomials of degree no greater than p and by $\pi_{p,q}$ the set of all rational functions $r_{p,q}(z) = \dfrac{P_p(z)}{P_q(z)}$ and $P_q(0) = 1$. Then a Padé (p,q)-approximation for e^{-z} is defined as an element $R_{p,q}(z) \in \pi_{p,q}$ such that

$$|e^{-z} - R_{p,q}(z)| = O(|z|^{p+q+1}) \text{ as } |z| \to 0.$$

It is well known that a Padé approximation for e^{-z} exists, is unique and is represented by the formula $R_{p,q}(z) = P_{p,q}(z)/Q_{p,q}(z)$, where

$$P_{p,q}(z) = \sum_{j=0}^{p}\frac{(p+q-j)!p!(-z)^j}{(p+q)!j!(p-j)!},\ Q_{p,q}(z) = \sum_{j=0}^{q}(p+q-j)!q!z^j(p+q)!j!(q-j)!.$$

In [174, 175], details of the location of poles and the order of convergence of rational approximations in different regions are given.

Definition 3.3. A rational approximation $r_{p,q}(\cdot) \in \pi_{p,q}$ for e^{-z} is said to be

(a) A-acceptable if $|r_{p,q}(z)| < 1$ for $\mathrm{Re}(z) > 0$;

(b) $A(\theta)$-acceptable if $|r_{p,q}(z)| < 1$ for $z \in \Sigma(\theta) = \{z : -\theta < \arg(z) < \theta, z \neq 0\}$.

It is well known that $R_{q,q}(z), R_{q-1,q}(z)$, and $R_{q-2,q}(z)$ are A-acceptable. But for $q \geq 3$ and $p = q-3$, the Padé functions are not A-acceptable.

Theorem 3.11 ([175]). *For any $q \geq 2$ and $p \geq 0$, the Padé approximation of e^{-z} has no poles in the sector*

$$S_{p,q} = \Big\{z : |\arg(z)| < \cos^{-1}\Big(\frac{q-p-2}{p+q}\Big)\Big\};$$

in particular, for $p \leq q \leq p+4$ all poles lie in the left half-plane.

Since $r(\cdot) \in \pi_{p,q}$ is an approximation of e^{-z}, it is natural to construct the operator-function $r(\tau_n A_n)^k$ which can be considered as an approximation of $\exp(tA_n)$ for $t = k\tau_n$. For simplicity, we assume in this section that $\|\exp(tA_n)\| \leq M, t \in \overline{\mathbb{R}}_+$.

Theorem 3.12 ([44]). *Let condition* (B) *be satisfied. There is a constant C depending on r such that if r is A-acceptable, then*

$$\|r(\tau_n A_n)^k\| \leq CM\sqrt{k} \text{ for } \tau_n > 0, k \in \mathbb{N}.$$

Remark 3.1. The term $\sqrt{k}$ in Theorem 3.12 cannot be removed in general; moreover, there are examples [55, 97], which show that the inequality $\|r(\tau_n A_n)^k\| \geq c\sqrt{k}, k \in \mathbb{N}$, holds.

We say that $r(\cdot) \in \pi_{p,q}$ is accurate of order $1 \leq d \leq p+q$ if $|e^{-z} - r(z)| = O(|z|^{d+1})$ as $|z| \to 0$.

Theorem 3.13 ([44]). *Let condition* (B) *be satisfied. Then there is a constant C depending on r such that, if r is A-acceptable and accurate of order d, then*

$$\|r(\tau_n A_n)^k u_n^0 - \exp(tA_n)u_n^0\| \leq CM\tau_n^d \|A_n^{d+1} u_n^0\| \text{ for } \tau_n > 0, k \in \mathbb{N}, u_n^0 \in D(A_n^{d+1}).$$

Theorem 3.14 ([44]). *Let condition (B_1) be satisfied. Then there is a constant C depending on r, such that if r is A-acceptable and accurate of order d, then*

$$\|r(\tau_n A_n)^k u_n^0 - \exp(tA_n)u_n^0\| \leq CM\tau_n^d \|A_n^d u_n^0\| \text{ for } \tau_n > 0, k \in \mathbb{N}, u_n^0 \in D(A_n^d).$$

Theorem 3.15 ([162, 185])**.** *Let condition* (B_1) *be satisfied. Then there is a constant* C *depending on* r*, such that if* r *is A-acceptable and accurate of order* d *with* $|r(\infty)| < 1$ *or condition* (3.6) *is satisfied, then*

$$\|r(\tau_n A_n)^k u_n^0 - \exp(tA_n)u_n^0\| \le CM\frac{\tau_n^\gamma}{t^{d-\gamma}}\|A_n^\gamma u_n^0\| \text{ for } \tau_n > 0, 0 \le \gamma \le d, t = k\tau_n, k \in \mathbb{N}.$$

In [54, 152, 154], the analogs of Theorems 3.13–3.15 were proved for multistep methods.

Let us recall that constant M_2 in condition (B_1), which defines α, $0 < \alpha < \dfrac{\pi}{2}$, by $M_2 \sin\alpha < 1$ [110] is such that

$$\|(\lambda I_n - A_n)^{-1}\| \le \frac{M}{|\lambda - \omega|} \text{ for any } \lambda \in \Sigma(\pi/2 + \alpha). \tag{3.8}$$

Theorem 3.16 ([55, 150])**.** *Let condition* (B_1) *be satisfied. Then there is a constant* C *depending on* r*, such that if* r *is* $A(\theta)$*-acceptable, accurate of order* d*, and* $\theta \in (\pi/2 - \alpha, \pi/2]$ *for* α *from condition* (3.8)*, then*

$$\|r(\tau_n A_n)^k\| \le CM \text{ for } \tau_n > 0, k \in \mathbb{N},$$

and

$$\|r(\tau_n A_n)^k - \exp(tA_n) - \gamma^k \exp(-\tau_n^{-b} a k_n (-A_n)^{-b})\| \le CM(k_n^{-d} + k_n^{-1/b}),\ t = k_n\tau_n,$$

where $\gamma = r(\infty)$ *and* a, b *are some positive constants.*

It is possible to show [151] that $\prod_{j=1}^k r(\tau_{n,j}A_n)$ is a stable approximation for $\exp(\sum_{j=1}^k \tau_{n,j}A)$ with a variable stepsize, but under condition $0 < c \le \tau_{n,i}/\tau_{n,j} \le C < \infty$, $i, j \in \mathbb{N}$.

3.3. Richardson's extrapolation method. Let us consider schemes (3.3) and (3.5) which have the order of convergence $O(\tau_n)$ and denote $\mathcal{U}_n^{\tau_n}(k_n) = U_n(t)u_n^0$ and $\overline{\mathcal{U}}_n^{\tau_n}(k_n) = \overline{U}_n(t)u_n^0$, $t = k_n\tau_n$. The following approach to the limit is valid.

Theorem 3.17 ([167])**.** *Assume that condition* (B) *is satisfied. Then for* $\overline{V}_n(t) = 2\overline{\mathcal{U}}_n^{\tau_n}(k_n) - \overline{\mathcal{U}}_n^{\tau_n/2}(2k_n)$*, one has*

$$\|\overline{V}_n(t) - u_n(t)\| \le \tau_n^2 M e^{\omega t} t^2 \|A_n^3 u_n^0\|,\ t = k_n\tau_n.$$

If, in addition, scheme (3.5) is stable, then for $V_n(t) = 2\mathcal{U}_n^{\tau_n}(k_n) - \mathcal{U}_n^{\tau_n/2}(2k_n), t = k_n\tau_n$*,*

$$\|V_n(t) - u_n(t)\| \le \tau_n^2 M e^{\omega t} t^2 \|A_n^3 u_n^0\|,\ t = k_n\tau_n.$$

Let us consider the Crank–Nicolson scheme

$$\frac{\tilde{U}_n(k\tau_n + \tau_n) - \tilde{U}_n(k\tau_n)}{\tau_n} = A_n \frac{\tilde{U}_n(k\tau_n + \tau_n) + \tilde{U}_n(k\tau_n)}{2},\ \tilde{U}_n(0) = I_n,\ k \in \mathbb{N}_0, \tag{3.9}$$

Theorem 3.18 ([167])**.** *Assume that condition* (B) *is satisfied and that scheme* (3.9) *is stable. Then* $\psi_n(t) = \frac{4}{3}\tilde{\mathcal{U}}_n^{\tau_n/2}(2k_n) - \frac{1}{3}\tilde{\mathcal{U}}_n^{\tau_n}(k_n)$ *satisfies*

$$\|\psi_n(t) - u_n(t)\| \leq c\tau_n^4 e^{\omega t} t^2 \|A_n^6 u_n^0\|,\ t = k_n\tau_n.$$

In general, we set $\mathcal{V}_n^{\tau_n}(t) = R_{p,q}(\tau_n A_n)^{k_n} u_n^0$, $t = k_n\tau_n$.

Theorem 3.19 ([167])**.** *Assume that condition* (B) *is satisfied,* $p = q$ *and the scheme which corresponds to* $\mathcal{V}_n^{\tau_n}$ *is stable. Then for* $\psi_n(t) = -\frac{1}{2^{2q}-1}\mathcal{V}_n^{\tau_n}(t) + \frac{2^{2q}}{2^{2q}-1}\mathcal{V}_n^{\tau_n/2}(t)$,

$$\|\psi_n(t) - u_n(t)\| \leq c\tau_n^{2q+2} e^{\omega t}\left(\frac{t^{3/2}}{\sqrt{\tau_n}}\|A_n^{2q+3}u_n^0\| + t^3\tau_n^{2q-3}\|A_n^{4q+2}u_n^0\|\right),\ t = k_n\tau_n.$$

Theorem 3.20 ([167])**.** *Assume that condition* (B_1) *is satisfied,* $p = q$ *and* $\tau_n\|A_n\| \leq \text{const}$. *Then for* $0 \leq \gamma \leq 2q$ *and* $\psi_n(t) = -\frac{1}{2^{2q}-1}\mathcal{V}_n^{\tau_n}(t) + \frac{2^{2q}}{2^{2q}-1}\mathcal{V}_n^{\tau_n/2}(t)$,

$$\|\psi_n(t) - u_n(t)\| \leq c\frac{\tau_n^{2q+2}}{t^{2q+2-\gamma}} e^{\omega t}\|A_n^{\gamma}u_n^0\|,\ t = k_n\tau_n.$$

3.4. Lax-type equivalence theorems with orders. The Lax equivalence theorem on the convergence of the solution of the approximation problem to the solution of the given well-posed Cauchy problem states that the stability of the method is necessary and sufficient for the convergence provided it is compatible. Recently, Lax's theorem with orders, which make it possible to consider "unstable" approximations, was obtained.

Definition 3.4. C_0-semigroups $\exp(tA_n)$ and $\exp(tA)$ are said to be *compatible of order* $O(\varphi(\tau_n))$ on a linear manifold $\mathcal{U} \subset E$ with respect to the semigroup $\exp(\cdot A)$ if $\exp(tA)\mathcal{U} \subseteq D(A)$ and there is a constant C such that

$$\|(A_n p_n - p_n A)\exp(tA)x\| \leq C\tau_n\varphi(\tau_n)e^{\omega t}|x|_{\mathcal{U}} \text{ for any } x \in \mathcal{U}, \tag{3.10}$$

where $|\cdot|$ denotes the seminorm on $\mathcal{U}$.

Definition 3.5. C_0-semigroups $\exp(tA_n)$ is said to be *stable of order* $O(M_n e^{\omega_n t})$ if there are constants M_n and ω_n such that

$$\|\exp(tA_n)\| \leq M_n e^{\omega_n t} \text{ for any } t \in \overline{\mathbb{R}}_+. \tag{3.11}$$

The following is a slight modification of [47–50] and [66–68], which was proved in [164].

Theorem 3.21. *Let a C_0-semigroup $\exp(\cdot A_n)$ be compatible of order $O(\varphi(\tau_n))$ on a linear manifold $\mathcal{U} \subset E$ with respect to a semigroup $\exp(\cdot A)$, $\exp(tA)\mathcal{U} \subset \mathcal{U}$, and let $|\exp(tA)x|_{\mathcal{U}} \leq M|x|_{\mathcal{U}}$. The following assertions are equivalent:*

(i) $\|\left(\exp(tA_n)p_n - p_n \exp(tA)\right)x\| \leq 2M_n e^{\omega_n t} K\left(\frac{C_n}{2} t\varphi(\tau_n), x; E, \mathcal{U}\right)$;

(ii) $\|\Big(\exp(tA_n)p_n - p_n \exp(tA)\Big)x\| \leq M_n e^{\omega_n t} \begin{cases} M_x, & x \in E, \\ \frac{C_n}{2} t\varphi(\tau_n)|x|_{\mathcal{U}}, & t = k_n\tau_n \in [0,T], x \in \mathcal{U}; \end{cases}$

(iii) $\|\exp(tA_n)\| \leq M_n e^{\omega_n t}$, $\|(A_n p_n - p_n A)\exp(tA)x\| \leq C_n \tau_n \varphi(\tau_n) e^{\omega t}|x|_{\mathcal{U}}$ *for any* $x \in \mathcal{U}, t \in \overline{\mathbb{R}}_+$,

where M_x is a constant depending only on x and $K(t, x; E, \mathcal{U}) = \inf\limits_{y \in \mathcal{U}} \Big\{\|x-y\|_E + t|y|_{\mathcal{U}}\Big\}$ is Peetre functional.

Definition 3.6. A family of discrete semigroups $\{U_n(k_n\tau_n)\}$ is said to be *compatible of order $O(\varphi(\tau_n))$* on a linear manifold $\mathcal{U} \subset E$ with respect to the semigroup $\exp(\cdot A)$ if $\overline{\mathcal{U}} = E$ and

$$\|(U_n(\tau_n)p_n - p_n \exp(\tau_n A))\exp(tA)x\| \leq C\tau_n\varphi(\tau_n)|x|_{\mathcal{U}} \text{ for any } x \in \mathcal{U}. \tag{3.12}$$

Theorem 3.22. *Let $\exp(tA_n)\mathcal{U}_n \subset \mathcal{U}_n$, let condition* (B) *hold, and let $|\exp(tA_n)x|_{\mathcal{U}_n} \leq Ce^{\omega t}|x_n|_{\mathcal{U}_n}$ for any $x_n \in \mathcal{U}_n$ and $t > 0$. Then the following conditions are equivalent:*

(a) $\|(U_n(k_n\tau_n) - \exp(k_n\tau_n A_n))x_n\| \leq M_n K\left(\frac{C_n k_n \tau_n}{2}\varphi(\tau_n), x_n, E_n, \mathcal{U}_n\right), n, k_n \in \mathbb{N}$;

(b) $\|(U_n(k_n\tau_n) - \exp(k_n\tau_n A_n))x_n\| \leq M_n \begin{cases} M_{x_n}, & x_n \in E_n, \\ \frac{C_n}{2} te^{t\omega}\varphi(\tau_n)|x_n|_{\mathcal{U}_n}, & x_n \in \mathcal{U}_n; \end{cases}$

(c) $\|U_n(k_n\tau_n)\|_{B(E_n)} \leq M_n$, $\|(U_n(\tau_n) - \exp(\tau_n A))\exp(tA_n)x_n\| \leq \frac{C_n M_n}{2}\tau_n e^{\omega t}\varphi(\tau_n)|x_n|_{\mathcal{U}_n}$, *where* $k_n\tau_n = t \in [0,T]$.

Definition 3.7. A family of discrete semigroups $\{U(k_n\tau_n)\}$ is said to be *stable* of order $O(1/\psi(n^{-1}))$ if

$$\|U_n(k_n\tau_n)\|_{B(E_n)} \leq C/\psi(n^{-1}) \text{ for } n, k_n \in \mathbb{N}, 0 < \tau_n \leq \tau^*, \tau_n k_n \in [0,T]. \tag{3.13}$$

Theorem 3.23. *Let a discrete semigroup $\{U(k_n\tau_n)\}$ be compatible of order $O(\varphi(\tau_n))$ on a linear manifold $\mathcal{U} \subset E$ with respect to the semigroup $\exp(\cdot A)$. The following assertions are equivalent:*

(i) $\|U_n(k_n\tau_n)\|_{B(E_n)} \leq C/\psi(n^{-1})$;

(ii) $\|(U_n(k_n\tau_n)p_n - p_n\exp(k_n\tau_n A))x\| \leq \frac{C}{\psi(n^{-1})}K(k_n\tau_n\varphi(\tau_n), x; E, \mathcal{U}), n, k_n \in \mathbb{N}$;

(iii)

$$\|(U_n(k_n\tau_n)p_n - p_n\exp(k_n\tau_n A))x\| \leq \frac{C}{\psi(n^{-1})}\begin{cases} M_x, & x \in E, \\ k_n\tau_n\varphi(\tau_n)|x|_{\mathcal{U}}, & k_n\tau_n \in [0,T], x \in \mathcal{U}, \end{cases}$$

where M_x is a constant depending only on x.

Theorem 3.24. *Let* $|\exp(tA)x|_{\mathcal{U}} \leq C|x|_{\mathcal{U}}$ *for any* $x \in \mathcal{U}$ *and* $t \in [0,T]$. *Then the following conditions are equivalent:*

(i) *The family of operators* $\{U(k_n\tau_n)\}$ *is compatible of order* $O(\varphi(\tau_n))$ *on a linear manifold* $\mathcal{U} \subset E$ *with respect to the semigroup* $\exp(\cdot A)$ *and stable of order* $O(1/\phi(n^{-1}))$;

(ii) $\|(U_n(k_n\tau_n)p_n - p_n \exp(k_n\tau_n A))x\| \leq \dfrac{C}{\psi(n^{-1})} K(k_n\tau_n\varphi(\tau_n), x; E, \mathcal{U}), n, k_n \in \mathbb{N}$;

(iii) $\|(U_n(k_n\tau_n)p_n - p_n \exp(tA))x\| \leq \dfrac{C}{\psi(n^{-1})} \begin{cases} M_x, & x \in E, \\ k_n\tau_n\varphi(\tau_n)|x|_{\mathcal{U}}, & t = k_n\tau_n \in [0,T],\ x \in \mathcal{U}. \end{cases}$

On an extension of Lax–Richtmyer theory see [157, 181].

For a particular case where $E = L^p(\mathbb{R}^d)$ and the operator $A \equiv P(D) = \sum_{|\alpha| \leq r} p_\alpha D^\alpha$ on E, one can consider the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = P(D)u(x,t),\ u(x,0) = u^0(x), x \in \overline{\mathbb{R}}_+, \tag{3.14}$$

with $P(D)$ such that (3.14) is well-posed in the sense $\|u(\cdot,t)\|_{L^p(\mathbb{R})} \leq c\,\|u^0(\cdot)\|_{L^p(\mathbb{R})}, t \in \overline{\mathbb{R}}_+$.

Let us denote $\hat{P}(\xi) = \sum_{|\alpha| \leq r} p_\alpha (i\xi)^\alpha$. It is well known that (3.14) is well-posed iff $\|\exp(t\hat{P})\|_{M_p} \leq C$, $t \in \overline{\mathbb{R}}_+$, where M_p is the space of Fourier multipliers.

The semidiscrete approximation of (3.14) is given by

$$\frac{\partial u_n(x,t)}{\partial t} = P_h(D)u_n(x,t),\ u_n(x,0) = u_n^0(x), x \in \overline{\mathbb{R}}_+, \tag{3.15}$$

where $P_h(D_h) = h^{-r} \sum_{|\alpha| \leq r} p_\alpha(h) \sum_{\beta \in I_\alpha} b_\beta u_n(x + \beta h, t)$ and $\hat{P}_h(\xi) = h^{-r} \sum_{|\alpha| \leq r} p_\alpha(h) \sum_{\beta \in I_\alpha} b_\beta e^{i\langle \xi, h\beta \rangle}$. The operator $P_h(D_h)$ is said to be compatible with the operator $P(D)$ of order μ if $\hat{P}_h(\xi) - \hat{P}(\xi) = h^\mu |\xi|^{r+\mu} Q(h\xi), r = \deg \hat{P}_h(\xi), Q$ is an infinitely differentiable function, and $|Q(\eta)| \geq Q_0 > 0$ for $0 < |\eta| \leq \epsilon_0$.

Theorem 3.25 ([43])**.** *Let* $P(D)$ *and* $P_h(D_h)$ *be compatible of order* μ, *and* (3.14) *and let* (3.15) *be well posed. Then for every* $T > 0$, *there exists* $C > 0$ *such that* $\|(e^{tP_h(D_h)} - \exp(tP(D))u^0\|_{L^2(\mathbb{R}^d)} \leq ch^\mu \|u^0\|_{W^{2,r+\mu}(\mathbb{R}^d)}$, *and for* $0 < s < r + \mu$,

$$\begin{aligned} \|(e^{tP_h(D_h)} - \exp(tP(D))u^0\|_{L^2(\mathbb{R}^d)} &\leq ch^{\frac{s\mu}{\mu+r}} \|u^0\|_{B_2^s}, \\ \|(e^{tP_h(D_h)} - \exp(tP(D)))u_n^0\|_{L^\infty(\mathbb{R}^d)} &\leq ch^{\frac{s\mu}{\mu+r}} \|u^0\|_{B_2^{d/2+s}}, \end{aligned}$$

where $B_p^\theta = B_{p,\infty}^\theta$ *is the Besov space.*

It is remarked in [32] that for a quite general case, $B_{p,q}^\theta = (L^p(\mathbb{R}), D(A))_{\theta,q}$.

If we consider a full discretization scheme for (3.14) in the form $L_h U_n^{k+1} = B_h U_n^k, k = 0, 1, 2, \ldots,$ where $L_h v = \sum_\beta a_\beta(h) v(x + \beta h)$ and $B_h v = \sum_\beta b_\beta(h) v(x + \beta h)$, then a discrete semigroup can be constructed as $U_n(k\tau_n) u_n^0 = \mathcal{F}^{-1}\left(\hat{U}_n^k(\xi)\hat{u}_n^0\right), \hat{U}_n(\xi) = \hat{B}_n(\xi)/\hat{L}_n(\xi), \hat{B}_n(\xi) = \sum_\beta a_\beta(h) e^{\langle \xi, \beta h\rangle}$ (the time step τ_n is connected with h by $\tau_n/h^r = \text{const}$). Such a finite-difference operator $U_n(k\tau_n)$ approximates (3.14) with order μ if $\hat{U}_n(\xi) = e^{\tau_n \hat{P}(\xi)} + O(h^{r+\mu} + |\xi|^{r+\mu})$ as $\xi, h \to 0$.

Theorem 3.26 ([43]). *Let (3.14) be well-posed and let $U_n(k\tau_n)$ be stable in $E = L^2(\mathbb{R}^d)$ and approximate (3.14) with order $\mu > 0$. Then for any $T > 0$, there is a constant $c > 0$ such that*

$$\|(U_n(t) - \exp(tP(D)))u_n^0\|_{L^2(\mathbb{R}^d)} \le ch^\mu \|u_n^0\|_{W^{2,r+\mu}(\mathbb{R}^d)},$$

and for $0 < s < r + \mu$

$$\begin{aligned}
\|(U_n(t) - \exp(tP(D)))u^0\|_{L^2(\mathbb{R}^d)} &\le ch^{\frac{s\mu}{\mu+r}} \|u^0\|_{B_2^s}, \\
\|(U_n(t) - \exp(tP(D)))u^0\|_{L^\infty(\mathbb{R}^d)} &\le ch^\mu \|u^0\|_{B_{2,1}^{d/2+\mu+r}}, \\
\|(U_n(t) - \exp(tP(D)))u^0\|_{L^\infty(\mathbb{R}^d)} &\le ch^{\frac{s\mu}{\mu+r}} \|u^0\|_{B_2^{d/2+s}}, \ t = k\tau_n \in [0, T].
\end{aligned}$$

Conversely, the order of convergence implies the smoothness of u_n^0; see [32, 43].

The time discretization of parabolic problems with memory by the backward Euler method was considered in [27]. The stability and error estimates take place in the Banach space framework, and the results are used for obtaining error estimates in the L_2 and maximum norms for piecewise-linear finite-element discretizations in two space dimensions.

4. Backward Cauchy Problem

In a Banach space E, let us consider the backward Cauchy problem:

$$\begin{aligned}
&v'(t) = Av(t), \ t \in [0, T], \\
&v(T) = v^T,
\end{aligned} \tag{4.1}$$

where the element $v(0)$ is unknown. At least in two important cases it is not a well-posed problem; namely, if A is unbounded and generates an analytic C_0-semigroup or if the C_0-semigroup $\exp(\cdot A)$ is compact. Indeed, in these situations, the problem $\exp(TA)x = v^T$ is ill posed [51, 100, 196] in the sense that the operator $\exp(-TA)$ is not bounded on E and, moreover, $D(\exp(-TA)) \ne E$ in general. This means that in general the Cauchy problem (4.1) has a solution only for some (but not every) initial data v^T and the solution $v(0)$, if it is exists, does not depend continuously on the initial data. After changes of variables, setting $v(\eta) = u(T-\eta)$, one can rewrite the problem (4.1) in the form

$$\begin{aligned} &u'(t) = -Au(t),\ t \in [0,T], \\ &u(0) = u^0, \end{aligned} \tag{4.2}$$

where $u^0 = v^T$ is given and $u(T)$ is the element to be found. In this section, we consider the approximation of (4.2) with operator A, generating an analytic C_0-semigroup.

Definition 4.1. A bounded linear operator $R_{\epsilon,T}$ on the space E is called a regularizator for the Cauchy problem (4.2) if for any $\delta > 0$ and any $u^0 \in E$ for which a solution of (4.2) exists, there exists $\epsilon = \epsilon(\delta) > 0$ such that $\epsilon(\delta) \to 0$ as $\delta \to 0$ and $\sup\limits_{\|u^\delta - u^0\| \le \delta} \|R_{\epsilon(\delta),T} u^\delta - \exp(-TA)u^0\| \to 0$ as $\delta \to 0$.

In [140], it is proved that for the existence of a linear regularizator of the problem (4.2) that commutes with operator A, it is necessary and sufficient that $-A$ generate C_ϵ-semigroups $\mathcal{S}_\epsilon(t)$, $0 \le t \le T$, such that C_ϵ strongly converges to the identity operator I as $\epsilon \to 0$.

There are many regularizators, which can be considered for problem (4.2). For example, in [165], it was shown that if $-A_n^2$ generates a cosine operator function, then the method of quasi-reversibility, which is given by the Cauchy problems

$$u'_{n,\alpha}(t) = -A_n u_{n,\alpha}(t) - \alpha A_n^2 u_{n,\alpha}(t),\ u_{n,\alpha}(0) = u_n^0,$$

is a regularization method for (4.2), and $\|u_{n,\alpha}(T) - p_n u(T)\| \le C\alpha\Big(\|u_n^0 - p_n u^0\|/\delta + \rho\Big)$, where $\alpha = \alpha(\delta) = 1/\Big(\log(1/\delta) - \log\log(1/\delta) - o(\log^{-1}(1/\delta))\Big)$. In this case $\mathcal{S}_\alpha(t) \equiv \exp(-tA)\exp(-\alpha T A^2)$ is a C_α-semigroup with $C_\alpha = \exp(-\alpha T A^2)$ and $C_\alpha \to I$ as $\alpha \to 0$. Moreover, the generator of this C_α-semigroup is $-A$.

It has been shown in [60] that the stochastic differential equation

$$\begin{aligned} &du_\alpha(t) = -Au_\alpha(t)dt - \alpha A u_\alpha(t) dw(t), \\ &u(0) = u^0, \end{aligned} \tag{4.3}$$

where $w(\cdot)$ is the standard one-dimensional Wiener process, yields a stochastic regularization of (4.2). Explicitly, the operator-function

$$U_\alpha(t)u^0 = \frac{1}{2\pi i}\int_\Gamma e^{-t\lambda - \alpha\left(w(t)-w(0)\right)\lambda - \frac{1}{2}\alpha^2\lambda^2|t|}(\lambda - A)^{-1}u^0 d\lambda,\ t > 0,$$

which represents a solution of (4.3) for any $u^0 \in \mathfrak{A}_c(A)$, possesses the following properties:

$$\lim_{\alpha\to 0}\|U_\alpha(T)u^0 - \exp(-TA)u^0\| = 0, \tag{4.4}$$

$$\|U_\alpha(t)\| \le \frac{c_1}{\alpha\sqrt{|t|}}\exp\left(c_2\frac{\sqrt{|t|}}{\alpha} + c_3|t|^{-\mu})^2\right) + b(\alpha,|t|) \text{ for any } \alpha > 0. \tag{4.5}$$

Here, the function $b(\alpha, t)$ is bounded in the parameters α and t and $\mathfrak{A}_c(A)$ is the set of entire vectors of the operator A. By virtue of the inequality

$$\|U_\alpha(T)u^\delta - \exp(-TA)u^0\| \leq \|U_\alpha(T)\| \, \|u^\delta - u^0\| + \|U_\alpha(T)u^0 - \exp(-TA)u^0\|, \tag{4.6}$$

this means that there is a dependence on $\alpha = \alpha(\delta)$ such that $U_\alpha(T)$ becomes a regularizator. The operator function $t \mapsto \exp\Big((T-t)A\Big)U_\alpha(T)$, $0 \leq t \leq T$, is a C_α-semigroup with $C_\alpha = \exp(TA)U_\alpha(T)$. One can see that $C_\alpha \to I$ as $\alpha \to 0$, and that the generator of this C_α-semigroup is $-A$.

4.1. C-semigroups and ill-posed problems. Let C be a bounded linear operator on the Banach space E, i.e., $C \in B(E)$, and let $T > 0$ be some finite number.

Definition 4.2 ([191])**.** A family of bounded operators $\{\mathcal{S}(t) : 0 \leq t < T\}$ is called a *local C-semigroup* on E if

(i) $\mathcal{S}(t+s)C = \mathcal{S}(t)\mathcal{S}(s)$ for $t, s, t+s \in [0, T)$;

(ii) $\mathcal{S}(0) = C$;

(iii) $\mathcal{S}(\cdot)$ is strongly continuous on $[0, T)$.

Clearly, $\mathcal{S}(\cdot)$ is a commutative family. A local C-semigroup is said to be nondegenerate if the condition $\mathcal{S}(t)x = 0$ for all $t \in (0, T)$ implies $x = 0$. It is seen from Definition 4.2 that a local C-semigroup is nondegenerate [63] if and only if C is injective, i.e., $\mathcal{N}(C) = \{0\}$. Concerning construction with $\mathcal{N}(C) \neq \{0\}$ see [112], [113]. It is very interesting question how to apply the case of noninjective C, i.e., degenerate C-semigroups, for ill-posed problems. Unfortunately this approach still is not realized.

Starting from now on, we will consider only the case where $C \in B(E)$ is an injective operator.

Definition 4.3. The generator of $\{\mathcal{S}(t) : 0 \leq t < T\}$ is defined as the limit

$$-Gx := C^{-1} \lim_{h \to 0+} \frac{1}{h}(\mathcal{S}(h)x - Cx), x \in D(G),$$

with the natural domain $D(G) := \{x \in E : \exists \lim\limits_{h \to 0+} \frac{1}{h}(\mathcal{S}(h)x - Cx) \in R(C)\}$.

Proposition 4.1 ([182])**.** *The operator G is closed, $R(C) \subseteq \overline{D(G)}$ and $C^{-1}GC = G$.*

We denote the C-semigroup $\mathcal{S}(\cdot)$ with the generator $-G$ by $\mathcal{S}(\cdot G)$. Next, let $\tau \in (0, T)$. We set

$$L_\tau(\lambda)x := \int_0^\tau e^{-\lambda t}\mathcal{S}(tG)x dt, \; x \in E, \lambda > 0. \tag{4.7}$$

This is the so-called *local Laplace transform* of $\mathcal{S}(\cdot G)$.

Proposition 4.2. *Let $\mathcal{S}(\cdot G)$ be a local C-semigroup and let $L_\tau(\cdot)$ be the local Laplace transform of $\mathcal{S}(\cdot G)$. Then, for any $x \in E$, one has $L_\tau(\lambda)x \in D(G)$ and*

$$(\lambda + G)L_\tau(\lambda)x = Cx - e^{-\lambda\tau}\mathcal{S}(\tau G)x \text{ for all } \tau \in [0, T) \text{ and } \lambda > 0. \tag{4.8}$$

In the case of local C-semigroups, the spectrum $\sigma(-G)$ can be located on the half-line $[0, \infty)$. Therefore, in this case the Laplace transform of the local C-semigroup does not exist in general, and we follow the ideas of [30, 182, 191]. The function $L_\tau(\lambda)$ with property (4.8) is called an asymptotic resolvent.

Theorem 4.1 ([182]). *Let A be a closed linear operator on E and let $C \in B(E)$ be injective.*

(i) *If the operator A is the generator of a local C-semigroup $\{S(t); 0 \le t < T\}$ on E, then there exists an asymptotic C-resolvent $L_\tau(\lambda)$ of $-A$ such that*

$$\left\| \frac{d^m}{d\lambda^m} L_\tau(\lambda)x \right\| \le M_\tau \frac{m!}{\lambda^{m+1}} \|x\|, \; x \in E, \tag{4.9}$$

with $0 \le m/\lambda \le \tau, \lambda > a, m \in \mathbb{N} \cup \{0\}$, and the operator A satisfies $C^{-1}AC = A$.

(ii) *If $-A$ has an asymptotic resolvent which satisfies* (4.9), *and $CD(A)$ is dense in $D(A)$, $D(C^{-1}AC) \subset D(A)$, i.e., $Cx \in D(A)$ and $ACx \in R(C)$ imply $x \in D(A)$, then the part A_0 of A in $E_0 := \overline{D(A)}$ generates a local C-semigroup on E_0 with C equal to $C_0 := C|_{E_0}$.*

In particular, under the assumption that $\overline{CD(A)} = E$, the operator $-A$ generates a local C-semigroup on E if and only if $C^{-1}AC = A$ and there exists an asymptotic C-resolvent satisfying (4.9). *In this case, A has a dense domain.*

Remark 4.1. An asymptotic C-resolvent $L_\tau(\lambda)$ of operator $-A$ is compact for some $\lambda \in \mathbb{C}$ (and then for any λ large enough) if and only if $\mathcal{S}(\cdot A)$ is compact and uniformly continuous in t. Indeed, if $\mathcal{S}(\cdot A)$ is compact then by (4.7) and [219], it follows that $L_\tau(\lambda)$ is compact. Conversely, taking derivative of $L_\tau(\lambda)$ in τ and using the fact that $\mathcal{S}(\cdot A)$ is uniformly continuous in t we have that $\mathcal{S}(\cdot A)$ is compact as the uniform limit of compact operators. This fact could be used in the approximation of semilinear equations in case of the C-semigroups approach (see Sec. 6).

Let us consider the abstract Cauchy problem, which is given by (4.2).

Definition 4.4. A function $u(\cdot)$ is called a solution of $(ACP; T, y)$ if $u(\cdot)$ is continuously differentiable in $t \in [0, T)$, $u(t) \in D(A)$ for all $0 \le t < T$, and $u(\cdot)$ satisfies (4.2). We denote by $(ACP; T, CD(A))$ the problem $(ACP; T, y)$ with $y \in CD(A)$.

Definition 4.5. The Cauchy problem $(ACP; T, CD(A))$ is said to be *generalized well-posed* if for every $y \in CD(A)$, there is a unique solution $u(\cdot; y)$ of $(ACP; T, y)$ such that $\|u(t; y)\| \le M(t)\|C^{-1}y\|$ for $0 \le t < T$ and $y \in CD(A)$, where the function $M(t)$ is bounded on every compact subinterval of $[0, T)$.

It should be stressed here that the generalized well-posedness in the sense of Definition 4.5 is more general than that in the case of the problem in (3.1). Moreover, we can state that this generalized well-posedness is a solvability condition of (4.2) for which a regularizator exists.

Theorem 4.2 ([182]). *Let C be a bounded linear injection on E, and let A be a closed linear operator. Then the following assertions are equivalent*:

(I) *The operator $-A$ is the generator of a local C-semigroup*;

(II) $C^{-1}AC = A$, *and the problem $v'(t) = -Av(t) + Cx$, $t \in [0, T), v(0) = 0$, has a unique solution for every $x \in E$.*

If either $\rho(A) \neq \emptyset$ or A has a dense domain, (I) *and* (II) *are also equivalent to*

(III) $C^{-1}AC = A$, *and the problem $(ACP; T, CD(A))$ is generalized well-posed. Moreover, $u(t; y) = C^{-1}\mathcal{S}(tA)y$, $t \in [0, T)$, is a unique solution for every initial value $y \in CD(A)$.*

Since local C-semigroups are regularizators of the ill-posed problem (4.2) it is very important to present the approximation theory of local C-semigroups.

4.2. Semidiscrete approximation theorem. Within the general discretization scheme, let us consider the semidiscrete approximation of the problem (4.2) in the Banach spaces E_n :

$$\begin{aligned} &u_n'(t) = -A_n u_n(t), \quad t \in [0, T), \\ &u_n(0) = u_n^0, \end{aligned} \tag{4.10}$$

where the operators $-A_n$ are generators of local C_n-semigroups which are compatible with the operator $-A$ and $u_n^0 \to u^0$. We understand compatibility in the sense of the general approximation scheme as the $\mathcal{PP}$-convergence of $C_n \to C$ and the $\mathcal{PP}$-convergence of resolvents $(\tilde{\lambda} I_n - A_n)^{-1} \to (\tilde{\lambda} I - A)^{-1}$ for some $\tilde{\lambda} \in \rho(A) \cap \rho(A_n)$. Recall that in our general case (4.1), such a $\tilde{\lambda}$ does exist, since conditions (A) and (B) from Theorem 3.1 are naturally assumed to be satisfied.

Theorem 4.3 ([214] (Theorem ABC-C)). *Under the assumption $\overline{CD(A^2)} = E$, the following conditions (A_c) together with (B_c) are equivalent to condition (C_c).*

(A_c) *Compatibility. $C_n \to C$ and operators A_n and A are compatible*;

(B_c) *Stability. For any $0 < \tau < T$ there is some constant M_τ independent of n such that*

$$\|\mathcal{S}(tA_n)\| \leq M_\tau \text{ for } 0 \leq t \leq \tau \text{ and } n \in \mathbb{N};$$

(C_c) *Convergence. For any $0 < \tau < T$, we have*

$$\max_{t \in [0,\tau]} \|\mathcal{S}(tA_n)x_n^0 - p_n\mathcal{S}(tA)x^0\| = 0, \text{ as } n \to \infty,$$

whenever $x_n^0 \to x^0$.

Remark 4.2. In the case of exponentially bounded C-semigroups [64, 65] we can trivially change condition (A_c) to the condition

(A') $C_n \to C$ and $(\tilde{\lambda} - A_n)^{-1}C_n \to (\tilde{\lambda} - A)^{-1}C$ for some $\tilde{\lambda} \in \mathbb{C}$;

see [224] for details. Since, the construction can be done just as with condition (A'), in this case, we do not need to assume that $(\tilde{\lambda} - A_n)^{-1} \to (\tilde{\lambda} - A)^{-1}$ for some $\tilde{\lambda}$.

Remark 4.3. We have set the condition $\overline{CD(A^2)} = E$ for simplicity. For the general case one obtains the convergence on the set $\overline{CD(A^2)}$. In the case of integrated semigroups, such situations have been well investigated; see, e.g., [33, 35]. Actually, the paper [33] is devoted to the following effect observed in the study of convergence of semigroups. Suppose we are given a sequence of uniformly bounded semigroups $\{\exp(tA_n), t \geq 0\}, n \geq 1$ ($\|\exp(tA_n)\| \leq M, t \in \overline{\mathbb{R}}_+$) acting on the Banach space E. Assume furthermore that the limit $\lim_{n\to\infty}(\lambda I - A_n)^{-1}x = S(\lambda)x$ exists for any $x \in E$. If

$$\overline{\mathcal{R}(S(\lambda))} = E \tag{4.11}$$

($\mathcal{R}(S(\lambda))$ is common for all $\lambda > 0$), the semigroups in question strongly converge, by the Trotter–Kato theorem. One can also show that if condition (4.11) is relaxed, the limit

$$\lim_{n\to\infty} \exp(tA_n)x \tag{4.12}$$

exists for all $x \in \overline{\mathcal{R}(S(\lambda))}$ (see e.g. [115], [70, p. 34], or [33, 35]). As observed by T.G. Kurtz [115] for any $x \in E$, there exists the limit

$$\lim_{n\to\infty} \int_0^t \exp(sA_n)x ds. \tag{4.13}$$

In general, however, one cannot expect that (4.12) holds for $x \notin \overline{\mathcal{R}(S(\lambda))}$. This effect is of course related to Arendt's theorem, or rather to the generation theorem for "absolutely continuous integrated semigroups" presented in [37].

Let us consider the following semidiscretization of problem (4.3) in Banach spaces E_n:

$$\begin{aligned} du_{n,\alpha}(t) &= -A_n u_{n,\alpha}(t)dt - \alpha A_n u_{n,\alpha}(t)dw(t), \\ u_{n,\alpha}(0) &= u_n^0, \end{aligned} \tag{4.14}$$

where $u_n^0 \to u^0$, the operators A_n generate analytic semigroups, and $\{(\Omega, \mathcal{F}, \mathbb{P}), w(t)\}$ is the standard one-dimensional Wiener process (Brownian motion). As usual, the symbol $\mathbb{E}[\cdot]$ denotes the mathematical expectation.

We emphasize that the situation where $\sigma(A_n)$, $\sigma(A) \subset \mathbb{C} \setminus \Sigma\left(\frac{3}{4}\pi\right)$ is considered.

Theorem 4.4 ([214]). *Let the conditions* (A) *and* (B_1) *of Theorem* 3.2 *be satisfied, and let* $\delta_n > 0$ *be a sequence which converges to* 0 *as* $n \to \infty$. *Then there exists a sequence* α_n *such that* $u_{n,\alpha_n}(t) \to u(t)$ *for every* $t \in [0,T]$ *as* $n \to \infty$. *Here* $u_{n,\alpha_n}(\cdot)$ *is a solution of* (4.14) *and* $u(\cdot)$ *is a solution of* (4.2) *with* $u^0 \in \mathfrak{A}_c(A)$. *The convergence is understood in the following sense*:

$$\sup_{\|u_n^0 - p_n u^0\| \le \delta_n} \|u_{n,\alpha_n}(t) - p_n u(t)\| \to 0, \quad \mathbb{P}\text{-almost surely as } \delta_n \to 0.$$

4.3. Approximation by discrete C-semigroups. Following Sec. 3, we denote by $\{T_n(\cdot)\}$ a family of discrete semigroups, on E_n, respectively, i.e., $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [t/\tau_n]$. We define the generator of $T_n(\cdot)$ by the formula $-A_n = \dfrac{1}{\tau_n}(T_n(\tau_n) - I_n)$ and consider the process $\tau_n \to 0$, $k_n \to \infty$, $n \to \infty$. We assume that $C_n \in B(E_n)$ is an injective operator such that $T_n C_n = C_n T_n$. The discrete C_n-semigroup $U_n(\cdot)$ is defined as $U_n(t) = T_n(t) C_n$. In this subsection we also assume that the dimension of each of the spaces E_n is finite, but $\dim(E_n) \to \infty$ as $n \to \infty$.

Theorem 4.5 ([214] (Theorem ABC-C-discr)). *Under condition* (A) *of Theorem* 3.1 *and the assumption* $\overline{CD(A^2)} = E$, *the following conditions* (A_{cd}) *and* (B_{cd}) *together are equivalent to condition* (C_{cd}).

(A_{cd}) *Compatibility.* $C_n \to C$, *the operators* A_n, A *are compatible, and* $A_n \in B(E_n)$, $n \in \mathbb{N}$;

(B_{cd}) *Stability. For any* $0 < \tau < T$, *there is some constant* M_τ, *independent of* n, *such that*

$$\|U_n(t)\| \le M_\tau \text{ for all } 0 \le t \le \tau < T \text{ and } n \in \mathbb{N}$$

is satisfied uniformly for any choice of $\{\tau_n\}$ *and* $\{k_n\}$ *as long as* $\tau_n \to 0$, *and* $k_n = [t/\tau_n]$;

(C_{cd}) *Convergence. For any* $0 < \tau < T$, $\max_{t \in [0,\tau]} \|U_n(t) x_n^0 - p_n \mathcal{S}(tA) x^0\| \to 0$ *as* $\tau_n \to 0, n \to \infty$, *whenever* $x_n^0 \to x^0$.

Theorem 4.6 ([214]). *Let the conditions* (A_c) *and* (B_c) *be satisfied. Assume that condition* (A) *of Theorem* 3.1 *and the assumption* $\overline{CD(A^2)} = E$ *are satisfied and also that* $\tau_n \|A_n^2 C_n^{-1}\| \le \dfrac{q}{M_\tau T}$ *with* $q < 1$. *Then*

$$\|U_n(t)\| \le M_\tau (1-q)^{-1} \text{ for } 0 \le t \le \tau < T \text{ and any } n \in \mathbb{N}$$

uniformly for any choice of $\{\tau_n\}$ *and* $\{k_n\}$ *with* $\tau_n \to 0$, *as long as* $k_n = \left[\dfrac{t}{\tau_n}\right]$. *Moreover, for any* $0 < \tau < T$, $\max_{t \in [0,\tau]} \|U_n(t) x_n^0 - p_n \mathcal{S}(tA) x^0\| \to 0$ *as* $\tau_n \to 0$, $n \to \infty$, *whenever* $x_n^0 \to x^0$.

Remark 4.4. In fact, the scheme $\overline{U}_n(t) \equiv (I + \tau_n A_n)^{-k_n} C_n$ with $t = k_n \tau_n$ can be constructed even under condition (3.4). Indeed, $\tau_n A_n = \tau_n \lambda A_n (\lambda I_n - A_n)^{-1} - \tau_n A_n^2 (\lambda I_n - A_n)^{-1}$, and by the choice of λ we can make the second term less than ϵ, and then by choosing τ_n appropriately for a fixed λ, we obtain $\|\tau_n A_n\| \le 2\epsilon$, so that the scheme $\overline{U}_n(\cdot)$ is well defined.

Remark 4.5. In contrast to the well-posed case, for ill-posed problems it looks that the implicit and explicit methods of discretization in time are not so different in the sense of stability advantages (compare with Theorems 3.5 and 3.6). Moreover, under condition (3.4), it follows from the identity

$$(I_n - \tau_n A_n)^{k_n} C_n = (I_n - \tau_n^2 A_n^2)^{k_n} (I_n + \tau_n A_n)^{-k_n} C_n$$

and inequality

$$\|(I_n \pm \tau_n^2 A_n^2)^{k_n}\| \le C e^{t\tau_n \|A_n^2\|}, t = k_n \tau_n,$$

that the stability properties of the implicit and explicit methods are the same.

There are a lot of stochastic finite-difference schemes which could be written for problem (4.14). For example, some of the simplest are

$$U_{n,\alpha}(t+\tau_n) - U_{n,\alpha}(t) = -\tau_n A_n U_{n,\alpha}(t) - \alpha\, \Delta w(t) A_n U_{n,\alpha}(t), \tag{4.15}$$

$$\bar{U}_{n,\alpha}(t+\tau_n) - \bar{U}_{n,\alpha}(t) = -\tau_n A_n \bar{U}_{n,\alpha}(t+\tau_n) - \alpha\, \Delta w(t) A_n \bar{U}_{n,\alpha}(t), \tag{4.16}$$

where $\Delta w(t) = \Big(w(t) - w(t-\tau_n)\Big)$, $t = k_n \tau_n$, and $U_{n,\alpha}(0) = \bar{U}_{n,\alpha}(0) = I_n$.

Theorem 4.7 ([214]). *Let the conditions* (A) *and* (B_1) *of Theorem* 3.2 *be satisfied. Assume that the stability conditions* (3.4) *and* $\tau_n \|A_n^2\| e^{c\|A_n\|} = O(1)$ *are fulfilled for some constant* $c > 0$. *Then for* $\alpha_n = \sqrt{\tau_n}$ *the scheme* (4.15) *has the stable behavior in the following sense*:

$$\eta_n := \sup_{n\in\mathbb{N}} \sup \left\{ \mathbb{E}\left[\left\| U_{n,\alpha_n}(t) u_n^0 - \exp\left(-tA_n + \alpha_n (w(t) - w(0)) A_n - \frac{t}{2}\alpha_n^2 A_n^2 \right) u_n^0 \right\| \right] : \|u_n^0\| \le 1 \right\} \to 0,$$

and converges in the following sense:

$$\mathbb{E}\left[\|U_{n,\alpha_n}(t) u_n^0 - p_n u(t)\| \right] \le C\sqrt{\tau_n} \|A \exp(-TA) u^0\| + \|u_{n,\alpha_n}(t) - p_n u_{\alpha_n}(t)\| + C\eta_n \|u_n^0\|, \ 0 < t \le T.$$

For the scheme $\overline{U}_{n,\alpha_n}(\cdot)$, similar notions are employed.

We can also study the convergence of more sophisticated numerical methods. For example, in [45], in order to approximate (4.14), the following Runge–Kutta scheme was considered:

$$\begin{aligned} Y_1 &= U_{n,\alpha}(t) + \sqrt{\tau_n}\alpha A_n U_{n,\alpha}(t), \\ U_{n,\alpha}(t+\tau_n) - U_{n,\alpha}(t) &= -\tau_n A_n U_{n,\alpha}(t) + \alpha \Delta w(t) A_n U_{n,\alpha}(t) \\ &+ \frac{\sqrt{\tau_n}}{2} \left(\frac{(\Delta w(t))^2}{\sqrt{\tau_n}} - 1 \right) (\alpha_n A_n Y_1 - \alpha A_n U_{n,\alpha}(t)). \end{aligned} \tag{4.17}$$

Thus, the solution can be written in the form

$$
\begin{aligned}
&U_{n,\alpha}(t+\tau_n)\\
&= (I_n - \tau_n A_n - \frac{\alpha^2\tau_n}{2}A_n^2)^{k_n}\Pi_{k=1}^{k_n}(I_n + Z_n^{-1}\alpha\Delta w(t)A_n + (Z_n^{-1}/2)\alpha^2\Delta w(t)^2 A_n^2)U_{n,\alpha}(0), \qquad (4.18)
\end{aligned}
$$

where $Z_n = \left(I_n - \tau_n A_n - \frac{\alpha^2\tau_n}{2}A_n^2\right)$.

Theorem 4.8 ([45]). *Let conditions* (A) *and* (B) *of Theorem* 3.2 *be satisfied. Assume that the stability conditions* (3.4) *and* $\tau_n\|A_n^2\|e^{c\|A_n\|} = O(1)$ *are fulfilled for some constant* $c > 0$. *Then for* $\alpha_n = \sqrt{\tau_n}$, *scheme* (4.18) *has the stable behavior in the following sense*:

$$
\eta_n := \sup\left\{\mathbb{E}\left[\left\|U_{n,\alpha_n}(t)u_n^0 - e^{\left(-tA_n+\alpha_n(w(t)-w(0))A_n-\frac{t}{2}\alpha_n^2A_n^2\right)}u_n^0\right\|\right] : \|u_n^0\| \le 1\right\} \to 0,
$$

and converges in the following sense:

$$
\begin{aligned}
&\mathbb{E}\left[\|U_{n,\alpha_n}(t)u_n^0 - p_n u(t)\|\right]\\
&\le C\sqrt{\tau_n}\|A\exp(-TA)u^0\| + \|u_{n,\alpha_n}(t) - p_n u_{\alpha_n}(t)\| + C\eta_n\|u_n^0\|, \quad 0 < t \le T.
\end{aligned}
$$

In the case of the well-posed problem

$$
\begin{cases}
du_\alpha(t) = Au_\alpha(t)dt + \alpha A u_\alpha(t)dw(t),\\
u(0) = u^0,
\end{cases} \qquad (4.19)
$$

where the operator A generates an analytic C_0-semigroup, the semidiscrete and full-discretization schemes do not need additional stability assumptions and the order of convergence will be defined just by the compatibility property of the scheme. More precisely, the term $e^{\lambda t}$ under the integral leads to the absolute convergence of the integral independently of the behavior of α on any compact set. For example, we have the following assertion.

Theorem 4.9 ([45]). *Let the conditions* (A) *and* (B'') *of Theorem* 3.2 *be satisfied. Assume that the stability condition* (3.4) *is fulfilled for some constant* $C > 0$. *Then for any* $\alpha_n \in [0, \alpha']$, *the scheme like* (4.18) *has the stable behavior in the following sense*:

$$
\sup\left\{\mathbb{E}\left[\left\|U_{n,\alpha_n}(t)u_n^0 - e^{\left(tA_n+\alpha_n(w(t)-w(0))A_n-\frac{t}{2}\alpha_n^2A_n^2\right)}u_n^0\right\|\right] : \|u_n^0\| \le 1\right\} \le \tau_n,
$$

and converges in the following sense:

$$
\mathbb{E}\left[\|U_{n,\alpha_n}(t)u_n^0 - p_n\exp(tA)u^0\|\right] \le C\alpha_n\|A\exp(tA)u^0\| + \|u_{n,\alpha_n}(t) - p_n u_{\alpha_n}(t)\| + C\tau_n\|u_n^0\|, \quad 0 < t \le T.
$$

5. Coercive Inequalities

In a Banach space E, let us consider the following inhomogeneous Cauchy problem:

$$\begin{aligned} &u'(t) = Au(t) + f(t),\ t \in [0,T], \\ &u(0) = u_0, \end{aligned} \tag{5.1}$$

where the operator A generates C_0-semigroup and $f(\cdot)$ is some function from $[0,T]$ into E. Problem (5.1) can be considered in various functional spaces. The most popular situations are the following settings: the well-posedness in $C([0,T];E), C^{\alpha,0}([0,T];E)$, and $L^p([0,T];E)$ spaces (see [9, 24, 139], and also the second article in this volume).

We say that problem (5.1) is well posed, say in $C([0,T];E)$, if, for any $f(\cdot) \in C([0,T];E)$ and any $u_0 \in D(A)$,

(i) problem (5.1) is uniquely solvable, i.e., there exists $u(\cdot)$ which satisfies the equation and boundary condition (5.1), $u(\cdot)$ is continuously differentiable on $[0,T]$, $u(t) \in D(A)$ for any $t \in [0,T]$ and $Au(\cdot)$ is continuous on $[0,T]$;

(ii) the operator $(f(\cdot), u^0) \to u(\cdot)$ as an operator from $C([0,T];E) \times D(A)$ to $C([0,T];E)$ is continuous.

In the case $u^0 \equiv 0$, the coercive well-posedness in $C([0,T];E)$ means that $\|Au(\cdot)\|_{C([0,T];E)} \leq c\|f(\cdot)\|_{C([0,T];E)}$. In general, the coercive well-posedness in the space $\Upsilon([0,T];E)$ for problem (5.1) means that it is well-posed in the space $\Upsilon([0,T];E)$ and

$$\|u'(\cdot)\|_{\Upsilon([0,T];E)} + \|Au(\cdot)\|_{\Upsilon([0,T];E)} \leq C\,(\|f(\cdot)\|_{\Upsilon([0,T];E)} + \|u^0\|_F),$$

where F is some subspace of E. For results of the coercive well-posedness see [9, 24, 139].

The semidiscrete approximation of (5.1) are the following Cauchy problems in Banach spaces E_n:

$$\begin{aligned} &u_n'(t) = A_n u_n(t) + f_n(t),\ t \in [0,T], \\ &u_n(0) = u_n^0, \end{aligned} \tag{5.2}$$

with operators A_n which generate C_0-semigroups, A_n and A are compatible, $u_n^0 \to u^0$ and $f_n \to f$ in appropriate sense. Following Sec. 3, it is natural to assume that conditions (A) and (B_1) are satisfied.

Here we are going to describe the discretization of (5.2) in time. The simplest difference scheme (Rothe scheme) is

$$\begin{aligned} &\frac{\overline{U}_n^k - \overline{U}_n^{k-1}}{\tau_n} = A_n \overline{U}_n^k + \varphi_n^k,\ k \in \left\{1, ..., \left[\frac{T}{\tau_n}\right]\right\}, \\ &\overline{U}_n^0 = u_n^0, \end{aligned} \tag{5.3}$$

where, for example, in the case of $f_n(\cdot) \in C([0,T];E_n)$, one can set $\varphi_n^k = f_n(k\tau_n), k \in \{1,...,K\}, K = \left[\dfrac{T}{\tau_n}\right]$, and in the case $f_n(\cdot) \in L^1([0,T];E_n)$, one can set

$$\varphi_n^k = \frac{1}{\tau_n} \int\limits_{t_{k-1}}^{t_k} f_n(s)ds,\ t_k = k\tau_n,\ k \in \{1,...,K\}.$$

5.1. Coercive inequality in $C_{\tau_n}([0,T];E_n)$ spaces. Denote by $C_{\tau_n}([0,T];E_n)$ the space of elements $\overline{\varphi}_n = \{\varphi_n^k\}_{k=0}^K$ such that $\varphi_n^k \in E_n, k \in \{0,...,K\}$, with the norm $\|\overline{\varphi}_n\|_{C_{\tau_n}([0,T];E_n)} = \max_{0\le k\le K} \|\varphi_n^k\|_{E_n}$.

We recall that coercive well-posedness in $C([0,T];E)$ implies [24] that A generates an analytic C_0-semigroup.

Theorem 5.1 ([24]). *Let condition (B_1) be satisfied. Then problem* (5.3) *is stable in the space $C_{\tau_n}([0,T];E_n)$, i.e.,*

$$\|\overline{U}_n\|_{C_{\tau_n}([0,T];E_n)} \le C\Big(\|\overline{\varphi}_n\|_{C_{\tau_n}([0,T];E_n)} + \|u_n^0\|\Big).$$

Theorem 5.2 ([24]). *Let condition (B_1) be satisfied. Then problem* (5.3) *is almost coercive stable in the space $C_{\tau_n}([0,T];E_n)$, i.e.,*

$$\|A_n\overline{U}_n\|_{C_{\tau_n}([0,T];E_n)} \le M\Big(\|A_nu_n^0\|_{E_n} + \min\Big(\log(1/\tau_n), 1 + \Big|\log\|A_n\|\Big|\Big)\|\overline{\varphi}_n\|_{C_{\tau_n}([0,T];E_n)}\Big).$$

It should be noted that if (5.1) is coercive well posed in the space $C([0,T];E)$, then [69] the operator A should be bounded or the space E should contain a subspace isomorphic to c_0. This means that problem (5.3) is not coercive well posed in $C_{\tau_n}([0,T];E_n)$ space in general.

For the explicit scheme

$$\begin{aligned} \frac{U_n^k - U_n^{k-1}}{\tau} &= A_nU_n^{k-1} + \varphi_n^k,\ k \in \{1,...,K\}, \\ U_n^0 &= u_n^0, \end{aligned} \tag{5.4}$$

Theorem 5.2 can be reconstructed, but under a stability condition.

Theorem 5.3 ([24]). *Let condition (B_1) is satisfied, and let $\tau_n \log\left(\dfrac{1}{\tau_n}\right)\|A_n\| \le \epsilon$ for sufficiently small $\epsilon > 0$. Then problem* (5.4) *is almost coercive stable in the space $C_{\tau_n}([0,T];E_n)$, i.e.,*

$$\|A_nU_n\|_{C_{\tau_n}([0,T];E_n)} + \|U_n\|_{C_{\tau_n}([0,T];E_{n,1-\frac{1}{\log\frac{1}{\tau_n}}})}$$

$$\le M\Big(\|A_nu_n^0\|_{E_{n,1-\frac{1}{\log\frac{1}{\tau_n}}}} + \min\Big(\log(1/\tau_n), 1 + \Big|\log\|A_n\|\Big|\Big)\|\overline{\varphi}_n\|_{C^{\tau_n}([0,T];E_n)}\Big),$$

where $\|u_n\|_{E_{n,\alpha}} = \Big(\int\limits_0^\infty \|A_n\exp(tA_n)u_n\|_{E_n}^{\frac{1}{1-\alpha}}dt\Big)^{1-\alpha}.$

Remark 5.1. The space $E_{n,\alpha}$ with equivalent norm coincides with the real interpolation space $(E_n, D(A_n))_{1-1/p,p}$; see [139].

5.2. Coercive inequality in $C^{\alpha,0}_{\tau_n}([0,T];E_n)$ spaces. Denote by $C^{\alpha,0}_{\tau_n}([0,T];E_n)$, $0<\alpha<1$, the space of the elements $\overline{\varphi}_n$ with the norm

$$\|\overline{\varphi}_n\|_{C^{\alpha,0}_{\tau_n}([0,T];E_n)} = \max_{0\le k\le K}\|\varphi_n^k\|_{E_n} + \max_{1\le k<k+l\le K}\|\varphi_n^{k+l}-\varphi_n^k\|_{E_n}(\tau_n k)^{\alpha}(l\tau_n)^{-\alpha}.$$

Theorem 5.4 ([183]). *Let condition (B_1) hold. Then the scheme (5.3) is coercive well-posed in $C^{\alpha,0}_{\tau_n}([0,T];E_n)$ with $0<\alpha<1$, i.e.,*

$$\|A_n\overline{U}_n\|_{C^{\alpha,0}_{\tau_n}([0,T];E_n)} \le \frac{M}{\alpha(1-\alpha)}\Big(\|A_n u_n^0\|_{E_n} + \|\overline{\varphi}_n\|_{C^{\alpha,0}_{\tau_n}([0,T];E_n)}\Big).$$

Roughly speaking, assumption (B_1) is necessary and sufficient for the coercive well-posedness in $C^{\alpha,0}_{\tau_n}([0,T];E_n)$ space.

5.3. Coercive inequality in $L^p_{\tau_n}([0,T];E_n)$ spaces. Denote by $L^p_{\tau_n}([0,T];E_n)$, $1\le p<\infty$, the space of elements $\overline{\varphi}_n$ with the norm

$$\|\overline{\varphi}_n\|_{L^p_{\tau_n}([0,T];E_n)} = \Big(\sum_{j=0}^{K}\|\varphi_n^k\|^p_{E_n}\tau_n\Big)^{1/p}.$$

Theorem 5.5 ([183]). *Let condition (B_1) hold. Let the difference scheme (5.3) be coercive well posed in $L^{p_0}_{\tau_n}([0,T];E_n)$ for some $1<p_0<\infty$. Then it is coercive well posed in $L^p_{\tau_n}([0,T];E_n)$ for any $1<p<\infty$ and*

$$\|A_n\overline{U}_n\|_{L^p_{\tau_n}([0,T];E_n)} + \max_{0\le k\le K}\|\overline{U}_n^k\|_{E_{n,1-1/p}} \le \frac{Mp^2}{p-1}\Big(\|\overline{\varphi}_n\|_{L^p_{\tau_n}([0,T];E_n)} + \|\overline{U}_n^0\|_{1-1/p}\Big).$$

It should be noted that in contrast to the case of $C^{\alpha,0}$-space, the analyticity of the semigroup $\exp(\cdot A)$ is not enough for the coercive well-posedness in L^p space [127], therefore, to state coercive well-posedness in L^p, we need some additional assumptions.

Theorem 5.6 ([183]). *Let $1<p,q<\infty, 0<\alpha<1$, and let condition (B_1) hold. Then the difference scheme (5.3) is coercive well posed in $L^p_{\tau_n}([0,T];E_{n,\alpha,q})$, i.e.,*

$$\|A_n\overline{U}_n\|_{L^p_{\tau_n}([0,T];E_{n,\alpha,q})} + \max_{0\le k\le K}\|U_n^k\|_{E_{n,1-1/p}} \le \frac{Mp^2}{(p-1)\alpha(1-\alpha)}\Big(\|\overline{\varphi}_n\|_{L^p_{\tau_n}([0,T];E_{n,\alpha,q})} + \|U_n^0\|_{1-1/p}\Big),$$

where $E_{n,\alpha,q}$ is the interpolation space $(E_n, D(A_n))_{\alpha,q}$ with the norm

$$\|u_n\|_{E_{n,\alpha,q}} = \Big(\int_0^\infty \|\lambda^\alpha A_n(\lambda I_n - A_n)^{-1}\|^q_{E_n}\frac{d\lambda}{\lambda}\Big)^{1/q}.$$

For the general Banach space E, we have the following results. Assume that A is the generator of the analytic semigroup $\exp(tA), t \in \mathbb{R}_+$, of linear bounded operators with an exponentially decreasing norm as $t \to \infty$. This means that stability condition (B_1'') holds with $\omega'' \leq 0$.

Theorem 5.7 ([23]). *Let condition* (B_1) *hold. Then the solution of difference scheme* (5.3) *is almost coercive stable, i.e.,*

$$\|A_n \overline{U}_n\|_{L^p_{\tau_n}([0,T];E_n)} \leq M\Big(\|A_n \overline{U}^0_n\|_{E_n} + \min\{\log \frac{1}{\tau_n},\ 1 + |\log \|A_n\|_{B(E_n)}|\}\ \|\varphi_n\|_{L^p_{\tau_n}([0,T];E_n)}\Big)$$

holds for any $p \geq 1$, *where* M *does not depend on* τ_n, u^0_n, *and* φ_n.

Of course, for schemes like

$$\begin{aligned} \frac{U^k_n - U^{k-1}_n}{\tau_n} &= A_n \left(\frac{U^k_n + U^{k-1}_n}{2}\right) + \varphi^k_n,\ n \in \{1, ..., K\}, \\ U^0_n &= u^0_n. \end{aligned} \tag{5.5}$$

the coercive well-posedness can be considered.

Theorem 5.8 ([23]). *Let condition* (B_1) *hold. Then the solution of difference scheme* (5.5) *is almost coercive stable, i.e., the estimate*

$$\Big\|\Big\{A_n \frac{U^j_n + U^{j-1}_n}{2}\Big\}\Big\|_{L^p_{\tau_n}([0,T];E_n)} \leq M\Big(\|A_n u^0_n\|_{E_n} + \min\Big\{\log \frac{1}{\tau_n}, 1 + |\log \|A_n\|_{E_n \mapsto E_n}|\Big\}\|\varphi_n\|_{L^p_{\tau_n}([0,T];E_n)}\Big)$$

holds for any $p \geq 1$, *where* M *does not depend on* τ_n, u^0_n, *and* φ_n.

Theorem 5.9 ([23]). *Let condition* (B_1) *hold and condition* (3.6) *be satisfied. Then the solution of difference scheme* (5.5) *is almost coercive stable, i.e., the estimate*

$$\|A_n U_n\|_{L^p_{\tau_n}([0,T];E_n)} \leq M\Big(\|A_n u^0_n\|_{E_n} + \min\Big\{\log \frac{1}{\tau_n}, 1 + |\log \|A_n\|_{E_n \mapsto E_n}|\Big\}\|\varphi_n\|_{L^p_{\tau_n}([0,T];E_n)}\Big)$$

holds for any $p \geq 1$, *where* M *does not depend on* τ_n, u^0_n, *and* φ_n.

The necessary and sufficient conditions for the coercive well-posedness of problem (5.1) in $L^p([0,T]; E)$ were obtained in [101, 220, 221]. More precisely, a Banach space E has the UMD property iff the Hilbert transform

$$Hf(t) = \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{t-s} f(s)ds$$

extends to a bounded operator on $L^p(\mathbb{R}; E)$ for some (all) $p \in (1, \infty)$. It is well known, that all subspaces and quotient spaces of $L^q(\Omega, \mu)$ with $1 < q < \infty$ have this property.

The Poisson semigroup on $L^1(\mathbb{R})$ is not coercive well posed on the $L^p(\mathbb{R}, E)$ space if E is not an UMD space (see [127]). Hence the assumptions on E to be an UMD space is necessary in some sense.

But it was an open problem whether every generator of an analytic semigroup on $L^q(\Omega, \mu)$, $1 < q < \infty$, provided the coercive well-posedness in $L^p(\mathbb{R}; E)$. Recently, Kalton and Lancien [103] gave a strong negative answer to this question. If every bounded analytic semigroup on a Banach space E is such that problem (5.1) is coercive well posed, then E is isomorphic to a Hilbert space.

If A generates a bounded analytic semigroup $\{\exp(zA) : |\arg(z)| \leq \delta\}$, on a Banach space E, then the following three sets are bounded in the operator norm:

(i) $\{\lambda(\lambda - A)^{-1} : \lambda \in i\mathbb{R}, \lambda \neq 0\}$;

(ii) $\{\exp(tA), tA\exp(tA) : t > 0\}$;

(iii) $\{\exp(zA) : |\arg z| \leq \delta\}$.

In Hilbert spaces, this already implies the coercive well-posedness in $L^p(\mathbb{R}_+; E)$, but only in Hilbert spaces E. The additional assumption that we need in more general Banach spaces E is the R-boundedness.

A set $\mathcal{T} \subset B(E)$ is said to be R-bounded if there is a constant $C < \infty$ such that for all $Z_1, \ldots, Z_k \in \mathcal{T}$ and $x_1, \ldots, x_k \in E, k \in \mathbb{N}$,

$$\int_0^1 \Big\| \sum_{j=0}^{k} r_j(u) Z_j(x_j) \Big\| du \leq C \int_0^1 \Big\| \sum_{j=0}^{k} r_j(u) x_j \Big\| du, \tag{5.6}$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables, e.g., the Rademacher functions $r_j(t) = \mathrm{sign}(\sin(2^j \pi t))$ on $[0, 1]$. The smallest C such that (5.6) is fulfilled, is called the R-boundedness constant of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$.

Theorem 5.10 ([221]). *Let A generate a bounded analytic semigroup* $\exp(tA)$ *on a UMD-space E. Then problem* (5.1) *is coercive well posed in the space $L^p(\mathbb{R}_+; E)$ if and only if one of the sets* (i), (ii) *or* (iii) *above is R-bounded.*

The interpretation of the discrete coercive inequality and a discrete semigroup defines the convolution operator of the form $\breve{A}_n \sum_{j=0}^{k} T_n^{k-j} Q_n \varphi_n \tau_n$ with some bounded operator $Q_n \in B(E_n)$, which usually has a smoothness property as it is clear from the proofs of Theorems 5.7 and 5.8. Here, $T_n(\tau_n)^k$ is a discrete semigroup, say, as in Sec. 3.1.

The boundedness of the convolution operator in $L^p_{\tau_n}(\mathbb{Z}_+; E_n)$ space implies the discrete coercive well-posedness in $L^p_{\tau_n}(\mathbb{Z}_+; E_n)$.

Also, in this section, we assume that Banach spaces E_n satisfy the collective UMD-property, i.e., we assume that the Hilbert transforms

$$H_n f_n(t) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{t-s} f_n(s) ds$$

extend to a bounded operators on $L^p(\mathbb{R}; E_n)$ for some (all) $p \in (1, \infty)$ such that all of them are bounded by a constant which does not depend on n. This assumption holds for example if all E_n can be embedded in a fixed space $L^p(\Omega)$ with $1 < p < \infty$.

Definition 5.1. A discrete semigroup $T_n(\cdot)$ with a generator $\breve{A}_n$ generates the coercive well-posedness on $L^p_{\tau_n}(\mathbb{Z}_+; E_n)$ space if the corresponding convolution operator $\varphi_n \mapsto \left\{\breve{A}_n \sum\limits_{j=0}^{k} T_n^{k-j} Q_n \varphi_n^j \tau_n\right\}$ is continuous on the $L^p_{\tau_n}(\mathbb{Z}_+; E_n)$ space.

Theorem 5.11 ([23]). *Assume that for convolution operator*

$$\varphi_n \mapsto \left\{\breve{A}_n \sum_{j=0}^{k} T_n^{k-j} Q_n \varphi_n^j \tau_n\right\},$$

the following conditions hold:

1^0 *the set* $\{\breve{A}_n(\lambda - T_n)^{-1} Q_n \tau_n : |\lambda| = 1, \lambda \neq 1, \lambda \neq -1\}$ *is R-bounded;*

2^0 *the set* $\{(\lambda - 1)(\lambda + 1)\breve{A}_n(\lambda - T_n)^{-2} Q_n \tau_n : |\lambda| = 1, \lambda \neq 1, \lambda \neq -1\}$ *is R-bounded.*

Then the discrete semigroup $T_n(\cdot)$ generates the coercive well-posedness on the $L^p_{\tau_n}(\mathbb{Z}_+; E_n)$ space.

Theorem 5.12 ([23]). *Let E_n be UMD Banach spaces. Also, assume that the set*

$$\{\lambda(\lambda - A_n)^{-1} : \lambda \in i\mathbb{R}, \lambda \neq 0\}$$

is R-bounded with the R-boundedness constant independent of n. Then the solution of difference scheme (5.3) *is coercive stable, i.e.,*

$$\|\breve{A}_n \overline{U}_n\|_{L^p_{\tau_n}(\mathbb{Z}_+; E_n)} \leq M \|\varphi_n\|_{L^p_{\tau_n}(\mathbb{Z}_+; E_n)} \tag{5.7}$$

holds for any $p \geq 1$, where M is independent of τ_n, u_n^0, and φ_n.

Remark 5.2. It should be noted that Theorem 3.2 can be reformulated in terms of R-boundedness with the change of condition (B_1) by the following condition: there is a $0 < \theta < \pi/2$ such that the set $\{\lambda(\lambda - A_n)^{-1} : \lambda \in \Sigma(\theta + \pi/2)\}$ is R-bounded with the R-boundedness constant independent of n. Condition (C_1) can be written, due to [221, Theorem 4.2], in the following form: $\exp(tA_n) \to \exp(tA)$ converges for any $t \in \mathbb{R}$ and there is $0 < \theta < \pi/2$ such that the set $\{\exp(zA_n) : z \in \Sigma(\theta)\}$ is R-bounded with the R-boundedness constant independent of n. Therefore, one of our assumption in Theorems 5.12 and 5.13 is in some sense condition (B_1) changed by the R-boundedness condition.

Theorem 5.13 ([23]). *Let E_n be UMD Banach spaces. Also, assume that the set $\{\lambda(\lambda - A)^{-1}\colon \lambda \in i\mathbb{R}, \lambda \neq 0\}$ is R-bounded with the R-boundedness constant independent of n. Then the solution of difference* *scheme* (5.5) *is coercive stable, i.e.,*

$$\left\| \left\{ \breve{A}_n \frac{U_n^k + U_n^{k-1}}{2} \right\} \right\|_{L^p_{\tau_n}([0,T];E_n)} \leq M \|\varphi_n\|_{L^p_{\tau_n}([0,T];E_n)} \tag{5.8}$$

holds for any $p \geq 1$, *where* M *does not depend on* τ_n, u_n^0, *and* φ_n.

Remark 5.3. Analyzing the proofs of Theorems 5.12 and 5.13, it is easy to see that one can set $\breve{A}_n = A_n$ in statements (5.7) and (5.8). Moreover, statement (5.8) can be written in the form

$$\|\breve{A}_n U_n\|_{L^p_{\tau_n}([0,T];E_n)} \leq M \|\varphi_n\|_{L^p_{\tau_n}([0,T];E_n)}.$$

The proof of this fact is based on the relation $\breve{A}_n = A_n(I_n - \frac{\tau_n}{2} A_n)^{-1}$.

It is possible to consider a more general Padé difference scheme [24] for $p = q-1$ or $p = q-2$. In this case, the difference scheme is written in the form

$$\frac{U_n^k - U_n^{k-1}}{\tau_n} = (\breve{A}_n U_n)^{k-1} + \varphi_n^{p,q,k},\ U_n^0 = u_n^0,\ 1 \leq k \leq K. \tag{5.9}$$

where $(\breve{A}_n U_n)^k = \left(\frac{R_{p,q}(\tau_n A_n) - I}{\tau_n} U_n \right)^{k-1}$ and $\|\varphi_n^k - \varphi_n^{p,q,k}\|_{E_n} \leq M \tau_n^{p+q}$. To formulate the coercive statements of Secs. 5.1–5.3, we just need to change the operator A_n by $\breve{A}_n$. We know from Theorem 3.16 that under condition (B_1) with $p = q$, the Padé approximation is stable, but, in general, it is not coercive stable. To obtain the coercive inequality, we need condition (3.6). Spaces where the problem considered can also be very different [24].

5.4. Coercive inequality in $B_{\tau_n}([0,T]; C_h^\theta(\Omega_h)) \cap C_h([0,T]; C_h(\bar{\Omega}_h))$**.** From the point of view of the numerical analysis, it is very interesting to consider problem (5.1) in the space $\Upsilon([0,T];E)$ such that E is smoother than $C(\Omega)$ (elements of such a space can easily be well approximated) and $\Upsilon([0,T];E)$ is like $C([0,T];E)$ or the space of bounded functions. An interesting fact is that such a situation is actually possible at least for a strongly elliptic operator of order 2 with coefficients of class $C^\theta(\overline{\Omega})$. Since operator $(p_n v)_i = v(ih)$ is very concrete in such space E, i.e., it takes values in the grid points, we omit p_n in the notation of this section.

Theorem 5.14 ([39])**.** *Let* Ω *be an open bounded subset of* $\mathbb{R}^d$ *lying to one side of its topological boundary* $\partial\Omega$, *which is a submanifold of* $\mathbb{R}^d$ *of dimension* $d-1$ *and class* $C^{2+\theta}$, *for some* $\theta \in (0,2) \setminus \{1\}$. *Let*

$$\mathcal{A} = \mathcal{A}(x, D_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) D_x^\alpha$$

be a strongly elliptic operator of order two (thus, $\mathrm{Re} \sum_{|\alpha|=2} a_\alpha(x) \xi^\alpha \geq \nu |\xi|^2$ *for some* $\nu > 0$ *and for any* $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^d$*) with coefficients of class* $C^\theta(\overline{\Omega})$. *Then there exist* $\mu \geq 0$ *and* $\phi_0 \in \left(\frac{\pi}{2}, \pi\right)$ *such that for* *any $\lambda \in \mathbb{C}$ with $|\lambda| \geq \mu$ and $|\operatorname{Arg}\lambda| \leq \phi_0$, the problem*

$$\lambda v - \mathcal{A}v = y,$$

$$\gamma_0 v = 0,$$

has a unique solution v belonging to $C^{2+\theta}(\overline{\Omega})$, for any $y \in C^\theta(\overline{\Omega})$ and for a certain $M > 0$,

$$|\lambda|^{1+\frac{\theta}{2}}\|v\|_{C(\overline{\Omega})} + |\lambda|\,\|v\|_{C^\theta(\overline{\Omega})} + \|v\|_{C^{2+\theta}(\overline{\Omega})} \leq M\Big(\|y\|_{C^\theta(\overline{\Omega})} + |\lambda|^{\frac{\theta}{2}}\|\gamma_0 y\|_{C(\partial\Omega)}\Big), \tag{5.10}$$

where γ_0 is the trace operator on $\partial\Omega$.

It is clear from (5.10) that the operator $\mathcal{A}$ does not generate a C_0-semigroup in $E = C^\theta(\overline{\Omega})$ space in general, but, following, say, [139], one can construct a semigroup $\exp(t\mathcal{A}), t \geq 0$, which is analytic.

Let $\mathcal{I} = \mathbb{Z}$, and let E be a Banach space with norm $\|\cdot\|$. For a grid function $U : \mathcal{I} \to E$, writing U_j or $(U)_j$ instead of $U(j)$ for any $j \in \mathcal{I}$, we set

$$B(\mathcal{I}; E) := \{U : \mathcal{I} \to E \,:\, \sup_{j\in\mathcal{I}} \|U_j\| < +\infty\},\ \|U\|_{B(\mathcal{I};E)} := \sup_{j\in\mathcal{I}} \|U_j\|.$$

It is easily seen that $B(\mathcal{I}; E)$ is a Banach space with the norm $\|\cdot\|_{B(\mathcal{I};E)}$. If the set $\mathcal{I}$ is some interval, say, $\mathcal{I} = (a, \infty)$, we denote by $B(\mathcal{I}; E)$ the set of all bounded functions from $\mathcal{I}$ into E.

For the grid function $U : \mathcal{I} \to E$ and $h > 0$, we define the operator ∂_h by formula

$$(\partial_h U)_j := h^{-1}(U_{j+1} - U_j).$$

For any $m \in \mathbb{Z}$ we set $(\partial_h^m U)_j := h^{-m} \sum\limits_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} (-1)^{m-i} U_{j+i}$. If $U \in B(\mathcal{I}; E)$, we set

$$\|U\|_{C_h^m(\mathcal{I};E)} := \max\Big\{\|\partial_h^r U\|_{B(\mathcal{I};E)} \,:\, 0 \leq r \leq m\Big\}.$$

Finally, let $\theta \in (0, 1)$. We define

$$[U]_{C_h^\theta(\mathcal{I};E)} := \sup\Big\{\Big((k-j)h\Big)^{-\theta}\|U_k - U_j\| \,:\, j, k \in \mathcal{I}, j < k\Big\},$$

and if $m \in \mathbb{N}_0$,

$$\|U\|_{C_h^{m+\theta}(\mathcal{I};E)} := \max\Big\{\|U\|_{C_h^m(\mathcal{I};E)}, [\partial_h^m U]_{C_h^\theta(\mathcal{I},E)}\Big\}.$$

In the same context, we denote by $B(\mathcal{I}; E)$ the space $C_h^0(\mathcal{I}; E)$. If $E = \mathbb{C}$, we write simply $B(\mathcal{I})$ or $C_h^0(\mathcal{I})$.

Let $f \in B(\mathbb{N}; E)$. We denote by $\tilde{f}$ the extension of f to $\mathbb{N}_0$ such that $\tilde{f}_0 = 0$. For a nonnegative real number ω, we define

$$\|f\|_{C_{h,0}^\omega(\mathbb{N};E)} := \|\tilde{f}\|_{C_h^\omega(\mathbb{N}_0;E)}. \tag{5.11}$$

Now let $L > 0, n \in \mathbb{N}, n \geq 3$, and let $h = \dfrac{L}{n}$. For $j \in \mathcal{I} := \{1, ..., n-1\}$, we are given complex numbers a_j, b_j, b'_j, and c_j satisfying the following conditions (ι):

($\iota 1$) there exists $\nu > 0$ such that $\mathrm{Re}(a_j) \geq \nu$ for every $j \in \mathcal{I}$;

($\iota 2$) $\max\Big\{|a_j|, |b_j|, |b'_j|, |c_j|\Big\} \leq Q$ for every $j \in \mathcal{I}$ with $Q > \nu$;

($\iota 3$) there exists $\omega : [0, L] \to [0, +\infty)$ such that $\omega(0) = 0$ and ω is continuous at 0 such that for $j, k \in \mathcal{I}$ with $j \leq k$,

$$|a_k - a_j| \leq \omega\Big((k-j)h\Big).$$

For $\lambda \in \mathbb{C}$, we study the following problem:

$$\begin{aligned} &\lambda U_j - a_j(\partial_h^2 U)_{j-1} - b_j(\partial_h U)_j - b'_j(\partial_h U)_{j-1} - c_j U_j = f_j \quad \text{for } j = 1, ..., n-1, \\ &U_0 = U_n = 0. \end{aligned} \tag{5.12}$$

For this purpose, we set $\overline{\mathcal{I}} := \{0, 1, ..., n-1, n\}$ and for $U \in B(\mathcal{I}; E)$, define

$$\tilde{U}_j = \begin{cases} U_j & \text{if } j \in \mathcal{I}, \\ 0 & \text{if } j \in \{0, n\}. \end{cases}$$

We introduce the operator A_h in $B(\mathcal{I}; E)$ defined by

$$(A_h U)_j := a_j(\partial_h^2 \tilde{U})_{j-1} + b_j(\partial_h \tilde{U})_j + b'_j(\partial_h \tilde{U})_{j-1} + c_j \tilde{U}_j \ \text{ for } j \in \mathcal{I}. \tag{5.13}$$

Further, we assume that

($\iota_\theta 1$) there exists $\nu > 0$ such that $\mathrm{Re}(a_j) \geq \nu$ for every $j \in \overline{\mathcal{I}}$;

($\iota_\theta 2$) $\max\Big\{\|a\|_{C_h^\theta(\overline{\mathcal{I}})}, \|b\|_{C_h^\theta(\overline{\mathcal{I}})}, \|b'\|_{C_h^\theta(\overline{\mathcal{I}})}, \|b\|_{C_h^\theta(\overline{\mathcal{I}})}\Big\} \leq Q$ with $Q > \nu$.

Proposition 5.1 ([95]). *Assume that assumptions (ι_θ) are satisfied for some $\theta \in (0, 2) \setminus \{1\}$. Fix $\phi_0 \in \Big[0, \pi - \arccos\Big(\frac{\nu}{Q}\Big)\Big)$. Then there exists $\mu_0 > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda| \geq \mu_0, |\mathrm{Arg}(\lambda)| \leq \phi_0\} \subseteq \rho(A_h)$, where A_h is the operator defined in (5.13). Moreover, for every $r \in [0, 2]$ there exists $c > 0$ depending only on L, ν, Q, and r such that for every $f \in B(\mathcal{I}; E)$ and any $F \in B(\overline{\mathcal{I}}; E)$ with $F|_{\mathcal{I}} = f$, one has*

$$\|(\lambda - A_h)^{-1} f\|_{C_{h,0}^{\theta+r}(\mathcal{I}; E)} \leq c|\lambda|^{\frac{r}{2}-1}\Big(\|F\|_{C_h^\theta(\overline{\mathcal{I}}; E)} + |\lambda|^{\frac{\theta}{2}} \max\{\|F_0\|, \|F_n\|\}\Big).$$

Let us consider the following mixed Cauchy–Dirichlet parabolic problem:

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, x) = \mathcal{A}u(t, x) + f(t, x), && t \in [0, T], x \in [0, L], \\ &u(t, x') = 0, && t \in [0, T], x' \in \{0, L\}, \\ &u(0, x) = 0, && x \in [0, L], \end{aligned} \tag{5.14}$$

where $\mathcal{A}$ is a second-order differential operator and $L > 0$. We say that problem (5.14) has a strict solution if there exists a continuous function $u(t, x)$ having the first derivative with respect to t and derivatives of order less than or equal to 2 with respect to x which are continuous up to boundary of $[0,T]\times[0,L]$, i.e. $u\in C^1\Big([0,T];C(\overline{\Omega})\Big)\cap C\Big([0,T];C^2(\overline{\Omega})\Big)$ and the equations in (5.14) are satisfied.

Theorem 5.15 ([91])**.** *Consider problem* (5.14) *under the following assumptions*:

(I) T *and* L *are positive real numbers*;

(II) $\theta\in(0,1)\setminus\left\{\dfrac{1}{2}\right\}$;

(III)

$$\mathcal{A}u(x)=a(x)\frac{\partial^2 u}{\partial x^2}(t,x)+b(x)\frac{\partial u}{\partial x}(t,x)+c(x)u(t,x),$$

with $a,b,c\in C^{2\theta}([0,L])$;

(IV) a *is real-valued and* $\min a=\nu>0$;

(V) $f\in C([0,T]\times[0,L])$, $t\to f(t,\cdot)\in B\Big([0,T];C^{2\theta}([0,L])\Big)$; $t\to f(t,0)$ *and* $t\to f(t,L)$ *belong to* $C^{\theta}([0,T])$; $f(0,0)=f(0,L)=0$.

Then there exists a unique strict solution $u(\cdot)$ *of problem* (5.14)*. Such a solution belongs to* $B\Big([0,T];C^{2+2\theta}([0,L])\Big)$ *and* $\dfrac{\partial u}{\partial t}\in B\Big([0,T];C^{2\theta}([0,L]\Big)$.

Now let $\mathcal{I}$ be a set which can depend on a positive parameter h and the Banach space $X_h=B(\mathcal{I})$. Next, we introduce a linear operator A_h in X_h depending on h. In each case, $\rho(A_h)$ contains

$$\{\lambda\in\mathbb{C}\setminus\{0\}:|\lambda|\geq R \text{ and } |\operatorname{Arg}(\lambda)|\leq\phi_0\},$$

where $R>0$ and $\phi_0\in(\frac{\pi}{2},\pi)$, and there exists $M>0$ such that

$$\|(\lambda-A_h)^{-1}\|_{L(X_h)}\leq M|\lambda|^{-1}$$

for λ in the specified subset of $\mathbb{C}$. Here, R,ϕ_0, and M are independent of h. Then we consider another set $\tilde{\mathcal{I}}$ such that $\mathcal{I}\subseteq\tilde{\mathcal{I}}$; we set $\tilde{X}_h:=B(\tilde{\mathcal{I}})$. We define an extension operator $\mathcal{E}_h$ from X_h to $\tilde{X}_h$: in all our concrete cases, this is the extension with zero. Next, for $\theta\in(0,1)$, we introduce the norms $\|\cdot\|_{2\theta,h}$ and $\|\cdot\|_{2+2\theta,h}$ in $\tilde{X}_h$. The first of these norms is connected with $\|\cdot\|_X$ and the operator A_h by the following property: there exist two positive constants c_1 and c_2 independent of h such that for every $U\in X_h$,

$$c_1\|\mathcal{E}_hU\|_{2\theta,h}\leq\|U\|_{(X_h,D(A_h))_\theta}\leq c_2\|\mathcal{E}_hU\|_{2\theta,h}.$$

Then, for every, h we consider the restriction operator $\mathcal{R}_h\in L(\tilde{X}_h,X_h)$ such that $\mathcal{R}_h\mathcal{E}_h=I_{X_h}$. Let us also introduce a seminorm p_h in $\tilde{X}_h$: in concrete cases, we have $p_h(U)=\|U|_{\tilde{\mathcal{I}}\setminus\mathcal{I}}\|_{B(\tilde{\mathcal{I}}\setminus\mathcal{I})}$. We assume that if $|\lambda|\geq R$ and $|\operatorname{Arg}(\lambda)|\leq\phi_0$, for every $G\in\tilde{X}_h$, then

$$|\lambda|\,\|\mathcal{E}_h(\lambda-A_h)^{-1}\mathcal{R}_hG\|_{2\theta,h}+\|\mathcal{E}_h(\lambda-A_h)^{-1}\mathcal{R}_hG\|_{2+2\theta,h}\leq M\Big(\|G\|_{2\theta,h}+|\lambda|^{\theta}p_h(G)\Big).$$

Another inequality we impose is the following. If $|\lambda| \geq R$, $|\operatorname{Arg}(\lambda)| \leq \phi_0$ and $G \in \tilde{X}_h$, then

$$\|A_h(\lambda - A_h)^{-1}\mathcal{R}_h G\|_{X_h} \leq M|\lambda|^{-\theta}\Big(\|G\|_{2\theta,h} + |\lambda|^{\theta} p_h(G)\Big).$$

Such an inequality can be easily deduced in each of our examples. In the formulation below, we remove the parameter h. In the case of the backward Euler scheme (5.3), we have

Theorem 5.16 ([95]). *Let X and $\tilde{X}$ be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_{\tilde{X}}$, respectively, and let $A \in B(X), \mathcal{E} \in B(X, \tilde{X})$, and $\mathcal{R} \in B(\tilde{X}, X)$ be such that $\mathcal{R}\mathcal{E} = I_X$. Assume, moreover, that $\theta \in (0,1)$ and $\|\cdot\|_{2\theta}$ and $\|\cdot\|_{2+2\theta}$ are norms in $\tilde{X}$, while p is a seminorm in the same space. Finally, assume that there exist $R > 0$, $\phi_0 \in \left(\frac{\pi}{2}, \pi\right)$, $M > 0$ such that the following conditions are satisfied:*

(a) *$\{\lambda \in \mathbb{C} : |\lambda| \geq R, |\operatorname{Arg}(\lambda)| \leq \phi_0\} \subseteq \rho(A)$ and, for λ in this set,*

$$\|(\lambda - A)^{-1}\|_{B(X)} \leq M(1 + |\lambda|)^{-1};$$

(b) *for every $F \in X$,*

$$\|\mathcal{E}F\|_{2\theta} \leq M\|F\|_{(X,D(A))_\theta};$$

(c) *for every $V \in \tilde{X}, \lambda \in \mathbb{C}$ with $|\lambda| \geq R$ and $|\operatorname{Arg}(\lambda)| \leq \phi_0$,*

$$\begin{aligned} &(1 + |\lambda|)^{-1}\|\mathcal{E}(\lambda - A)^{-1}\mathcal{R}V\|_{2\theta} + \|\mathcal{E}(\lambda - A)^{-1}\mathcal{R}V\|_{2+2\theta} \\ &+(1 + |\lambda|)^{\theta}\|A(\lambda - A)^{-1}\mathcal{R}V\|_X \leq M\Big(\|V\|_{2\theta} + (1 + |\lambda|)^{\theta} p(V)\Big); \end{aligned}$$

(d) *$p(V) \leq \|V\|_{2\theta}$ for every $V \in \tilde{X}$ and $p(\mathcal{E}F) = 0$ for every $F \in X$;*

(e) *$\|\mathcal{R}V\|_X \leq \|V\|_{2\theta}$ for every $V \in \tilde{X}$.*

Let $T > 0$, $K \in \mathbb{N}, K \geq 2$, and $\tau = \dfrac{T}{K}$. Assume that $\tau R < 1$.

Let $\overline{G} \in B(\{0,1,...,K\}; \tilde{X})$ be such that $G^0 = 0$; consider problem (5.3) with $\varphi^k = \mathcal{R}G^k$ for $k = 1,...,K$ and $U^0 = 0$. Then, for $\overline{U} \in B(\{0,1,...,K\}; X)$ which solves (5.3), one has

$$\|\mathcal{E}\overline{U}^k\|_{2+2\theta} \leq c\Big(\max_{0 \leq k \leq K}\|G^k\|_{2\theta} + \max_{0 \leq k_1 < k_2 \leq K}((k_2 - k_1)\tau)^{-\theta} p(G^{k_2} - G^{k_1})\Big), \tag{5.15}$$

for $k = 0,1,...,K$, where c is a positive constant depending only on θ, R, ϕ_0, M, and T and is independent of τ_n and $\overline{G}$.

We now consider the Crank–Nicolson scheme: we replace (5.3) by (5.5). Theorem 5.16 has the following analog.

Theorem 5.17 ([94]). *Assume that the assumptions of Theorem 5.16 are satisfied and, moreover,*

(f) *$\|\tau A\|_{B(X)} \leq S$ with some $S > 0$;*

(g) *if* $|\lambda| \geq 2S$, *then*

$$\|\mathcal{E}(\lambda - \tau A)^{-1}\mathcal{R}V\|_{2\theta} \leq M\left(\|V\|_{2\theta} + \tau^{-\theta}p(V)\right) \text{ for every } V \in \tilde{X};$$

(h) $\|\mathcal{E}\mathcal{R}V\|_{2\theta} \leq M\Big(\|V\|_{2\theta} + \tau^{-\theta}p(V)\Big)$ *for every* $V \in \tilde{X}$;

(i) $2\tau R < 1$.

Let $\overline{G} \in B(\{0,1,...,K\};\tilde{X})$ *be such that* $G^0 = 0$; *consider problem* (5.5) *with* $\varphi^k = \mathcal{R}G^k$ *for* $k = 1,...,K$ *and* $U^0 = 0$. *If* $U \in B\Big(\{0,1,...,K\};X\Big)$ *solves* (5.5) *for* $k = 0,1,...,K$, *then*

$$\|\mathcal{E}U^k\|_{2+2\theta} \leq c\Big(\max_{0\leq k\leq K}\|G^k\|_{2\theta} + \max_{0\leq k_1<k_2\leq K}((k_2-k_1)\tau)^{-\theta}p(G^{k_2}-G^{k_1})\Big), \tag{5.16}$$

where c *is a positive constant depending only on* θ, R, ϕ_0, M, S, *and* T *and is independent of* τ_n *and* $\overline{G}$.

An application of Theorems 5.16 and 5.17 to the discretization of problem (5.14) is the following. Let $K, n \in \mathbb{N}$. We set $\tau := \dfrac{T}{K}$ and $h := \dfrac{L}{n}$. We assume that $K \geq 2$ and $n \geq 3$. For $j = 0,1,...,n$, we set $a_j := a(jh)$, $b_j = \dfrac{1}{2}b(jh)$, $c_j := c(jh)$, $\mathbf{N}_{n-1} := \{1,...,n-1\}$, $\overline{\mathbf{N}}_n := \{0,1,...,n-1,n\}$, and

$$X := B(\mathbf{N}_{n-1}), \tilde{X} := B(\overline{\mathbf{N}}_n). \tag{5.17}$$

If $V \in X$ as before, for $i \in \mathbf{N}_{n-1}$, we set

$$(A_hV)_i := a_i\frac{\tilde{V}_{i+1} - 2\tilde{V}_i + \tilde{V}_{i+1}}{h^2} + b_i\frac{\tilde{V}_{i+1} - \tilde{V}_{i-1}}{2h} + c_i\tilde{V}_i, \tag{5.18}$$

where

$$\tilde{V}_i = (\mathcal{E}V)_i = \begin{cases} V_i & \text{if } 1 \leq i \leq n-1, \\ 0 & \text{if } i \in \{0, n\}. \end{cases}$$

Next, we define

$$\mathcal{R} \in B(\tilde{X}, X), \mathcal{R}V := V|_{\mathbf{N}_{n-1}} \tag{5.19}$$

for every $V \in \tilde{X}$. Then, again for $V \in \tilde{X}$ and $\theta \in \left(0, \dfrac{1}{2}\right)$, we set

$$\|V\|_{2\theta} := \max\{\|V\|_{\tilde{X}}, \max_{0\leq i_1<i_2\leq n}((i_2-i_1)h)^{-2\theta}|V_{i2} - V_{i_1}|\}, \tag{5.20}$$

$$\begin{aligned}\|V\|_{2+2\theta} := \max\{&\|V\|_{\tilde{X}}, \max_{0\leq i\leq n-1}|(\partial_hV)_i|, \max_{0\leq i\leq n-2}|(\partial_h^2V)_i|,\\ &\max_{0\leq i_1<i_2\leq n-2}((i_2-i_1)h)^{-2\theta}|(\partial_h^2V)_{i2} - (\partial_h^2V)_{i_1}|\},\end{aligned} \tag{5.21}$$

with

$$(\partial_hV)_i := \frac{V_{i+1} - V_i}{h} \text{ for } 0 \leq i \leq n-1, (\partial_h^2V)_i := \frac{V_{i+2} - 2V_{i+1} + V_i}{h} \text{ for } 0 \leq i \leq n-2,$$

$$p(V) := \max\{|V_0|, |V_n|\}. \tag{5.22}$$

One has the following result.

Theorem 5.18 ([94]). *With the notation* (5.17), (5.18), (5.19) *and* (5.20), *the assumptions* (a)–(e) *of Theorem* 5.16 *are satisfied, with* R, ϕ_0, M *independent of* h. *If we impose the further condition*

$$\tau_n \leq \alpha h^2, \tag{5.23}$$

the same also holds for assumptions (f)–(h) *of Theorem* 5.17 (*even with* S *independent of* n).

As a consequence, we have the following theorem.

Theorem 5.19 ([95]). *Consider the problem* (5.14) *under the assumptions of Theorem* 5.15. *With the notation* (5.17), (5.18), (5.19), *and* (5.20), *set* $G_j^k := f(k\tau_n, jh)$ *for* $k \in \{1, ..., K\}$, $j = 0, ..., n$. *Set* $\varphi^k := \mathcal{R}G^k$ *and denote by* G^0 *the zero in* $B(\overline{\mathbf{N}}_n)$.

Then if τ_n *is sufficiently small, the problem*

$$\begin{aligned} &\frac{\tilde{U}_j^k - \tilde{U}_j^{k-1}}{\tau_n} = a_i \frac{\tilde{U}_{i+1}^k - 2\tilde{U}_i^k + \tilde{U}_{i-1}^k}{h^2} + b_i \frac{\tilde{U}_{i+1}^k - \tilde{U}_{i-1}^k}{2h} + c_i \tilde{U}_i^k + \varphi_j^k, \\ &\tilde{U}_j^0 = 0, \end{aligned} \tag{5.24}$$

for $j \in 1, \ldots, n-1$, $k \in \{1, ..., K\}$ *has a unique solution such that*

$$\|\tilde{U}^k\|_{C_h^{2+2\theta}(\overline{\mathbf{N}}_n)} \leq c\Big(\|f\|_{B([0,T];C^{2\theta}([0,L]))} + \max\{\|f(\cdot, 0)\|_{C^\theta([0,T])}, \|f(\cdot, L)\|_{C^\theta([0,T])}\}\Big) \tag{5.25}$$

with c *independent of* h *and* τ_n.

An analogous result holds for the Crank–Nicholson scheme (5.5). Then we set $G_j^k := f\Big(\Big(k - \dfrac{1}{2}\Big)\tau_n, jh\Big)$ under the further condition (5.23).

Remark 5.4. It follows from Theorem 5.18 that, for scheme (5.5) with $u_n^0 = 0$ and under condition (5.23)

$$\|A_h U\|_{C_h^{2\theta}(\mathbf{N}_{n-1})} \leq c\Big(\|f\|_{B([0,T];C^{2\theta}([0,L]))} + \max\{\|f_n(\cdot, 0)\|_{C^\theta([0,T])}, \|f(\cdot, L)\|_{C^\theta([0,T])}\}\Big), \tag{5.26}$$

with c independent of h and τ_n. In the quoted papers Theorems 5.16 and 5.17 are also applied to the discretization of the heat equation in a square.

A counterexample in [94] and [93] shows that condition (5.23) cannot be removed in general. Finally, estimates of the order of convergence are given in [93].

The coercive inequalities and their discrete analogs in the spaces C and L^p for elliptic problems of the form, e.g.,

$$u''(t) = Au(t) + f(t), \quad u(0) = 0, \quad u(T) = u^T,$$

have been considered in [20, 21].

6. Approximations of Semilinear Equations

In a Banach space E, let us consider the semilinear Cauchy problem

$$u'(t) = Au(t) + f(t, u(t)), \ \ u(0) = u^0, \tag{6.1}$$

with the operator A, generating an analytic C_0-semigroup of type $\omega(A) < 0$, where the function f is smooth enough. The existence and uniqueness of solution of problem (6.1) have been discussed, e.g., in [9, 31, 96, 98, 139].

6.1. Approximations of Cauchy problem. By a semidiscrete approximation of problem (6.1), we mean the following Cauchy problems in the Banach spaces E_n:

$$u_n'(t) = A_n u_n(t) + f_n(t, u_n(t)), \ \ u_n(0) = u_n^0, \tag{6.2}$$

where the operators A_n generate analytic semigroups in E_n, A_n and A are compatible, the functions f_n approximate f and $u_n^0 \to u^0$.

Let Ω be an open set in a Banach space F, and let $\mathcal{B} : \bar{\Omega} \to F$ be a compact operator having no fixed points on the boundary of Ω. Then for the vector field $\mathcal{F}(x) = x - \mathcal{B}x$, the rotation $\gamma(I - \mathcal{B}; \partial\Omega)$ is defined; it is an integer-valued characteristic of this field. Let x^* be a unique isolated fixed point of the operator $\mathcal{B}$ in the ball S_{r_0} of radius r_0 centered at x^*. Then $\gamma(I - \mathcal{B}; \partial S_r) = \gamma(I - \mathcal{B}; \partial S_{r_0})$ for $0 < r \leq r_0$, and this common value of the rotations is called the index of the fixed point x^* and is denoted by $\operatorname{ind} x^*$.

Theorem 6.1 ([166])**.** *Assume that conditions* (A) *and* (B_1) *hold and compact resolvents* $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ *converge*: $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ *compactly for some* $\lambda \in \rho(A)$ *and* $u_n^0 \to u^0$. *Assume that*

(i) *the functions* f_n *and* f *are bounded and sufficiently smooth, so that there exists a unique mild solution* $u^*(\cdot)$ *of the problem* (6.1) *on* $[0, T]$ (*in this situation* $\operatorname{ind} u^*(\cdot) = 1$);

(ii) $f_n(t, x_n) \to f(t, x)$ *uniformly with respect to* $t \in [0, T]$ *as* $x_n \to x$;

(iii) *the space* E *is separable.*

Then for almost all n, *problems* (6.2) *have mild solutions* $u_n^*(t)$, $t \in [0, T]$, *in a neighborhood of* $p_n u^*(\cdot)$. *Each sequence* $\{u_n^*(t)\}$ *is* $\mathcal{P}$*-compact and* $u_n^*(t) \to u^*(t)$ *uniformly with respect to* $t \in [0, T]$.

Let us consider the time discretization with respect to the explicit difference scheme:

$$\frac{U_n(t + \tau_n) - U_n(t)}{\tau_n} = A_n U_n(t) + f_n(t, U_n(t)), \, U_n(0) = u_n^0, \, t = k\tau_n, k = \{0, \cdots, K\}. \tag{6.3}$$

Theorem 6.2 ([166]). *Assume that the conditions of Theorem* 6.1 *and condition* (3.6) *are satisfied. Then the functions* $U_n(t)$ *from* (6.3) *give an approximate mild solution* $u^*(\cdot)$ *of problem* (6.1) *and, moreover,* $U_n(t) \to u^*(t)$ *uniformly with respect to* $t \in [0,T]$.

Let us define the operator $\Re(u_n)(t) = u_n(t) - \int_0^t \exp((t-s)A_n) f(s, u_n(s)) ds$.

Remark 6.1. If we assume that the conditions of Theorem 6.1 hold and the functions $f(\cdot)$ and $f_n(\cdot)$ have Fréchet derivatives in some balls containing the solutions u^* and u_n^* and, moreover, assume that $f'_{n_u}(t, p_n u^*(t))$ are uniformly continuous with respect to the first and second arguments and $f'_{n u_n}(t, u_n(t)) \to f'_u(t, u^*(t))$ uniformly with respect to $t \in [0,T]$ for $u_n \to u^*$, then [166] for almost all n the problems (6.2) have mild solutions $u_n^*(t)$, $t \in [0,T]$, in the neighborhood of $p_n u^*(\cdot)$. Each sequence $\{u_n^*(\cdot)\}$ is $\mathcal{P}$-compact and $u_n^*(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$ and, moreover, for sufficiently large $n \geq n_0$ and some $T^* \leq T$ we have

$$c_1 \epsilon_n(u^*, u_n^0) \leq \|u_n^* - p_n u^*\|_{F_n} \leq c_2 \epsilon_n(u^*, u_n^0),$$

where the constants c_1 and c_2 are independent of n, $F_n = C([0,T]; E_n)$, and

$$\epsilon_n(u^*, u_n^0) = \max_{t \in [0,T^*]} \|\Re(p_n u^*)(t) - \exp(tA_n) u_n^0\|_{E_n}.$$

Let $\mathcal{U}_n(t) = (I_n + \tau_n A_n)^k$ and $\Im_n(u_n)(t) = u_n(t) - \sum_{l=1}^{k-1} \mathcal{U}_n((k-l)\tau_n) f_n(l\tau_n, u_n(l\tau_n))\tau_n$.

Remark 6.2. If we assume that the conditions of Theorem 6.2 hold and the functions $f(\cdot)$ and $f_n(\cdot)$ have Fréchet derivatives in some balls containing the solutions $u^*(\cdot)$ and $u_n^*(\cdot)$ and, moreover, assume that $f'_{n u_n}(t, p_n u^*(t))$ are uniformly continuous with respect to the first and second arguments and $f'_{n u_n}(t, u_n(t)) \to f'_u(t, u^*(t))$ uniformly with respect to $t \in [0,T]$ as $u_n \to u^*$ and condition (3.6) holds, then [166] the functions $U_n(t)$ from (6.3) give an approximate mild solution of the problem (6.1) and $U_n^*(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$ and, moreover, for sufficiently large $n \geq n_0$ and some $T^* \leq T$, we have

$$c_1 \epsilon_n(u^*, u_n^0) \leq \|U_n^* - p_n u^*\|_{F_n^{\tau_n}} \leq c_2 \epsilon_n(u^*, u_n^0),$$

where the constants c_1 and c_2 are independent of n, $F_n^{\tau_n} = \{u_n(k\tau_n) : \max_{0 \leq k\tau_n \leq T} \|u_n(k\tau_n)\|_{E_n} < \infty\}$ and $\epsilon_n(u^*, u_n^0) = \max_{t \in [0,T^*]} \|\Im_n(p_n u^*)(t) - \mathcal{U}_n(t) u_n^0\|_{E_n}$.

Schemes which have higher order of convergence than (6.3) are considered in [146, 166]. The Runge–Kutta methods for semilinear equations were considered in [79], [135–137, 146, 149].

6.2. Approximation of periodic problem. In a Banach space E, let us consider the semilinear T-periodic problem

$$v'(t) = Av(t) + f\big(t, v(t)\big), \ \ v(t) = v(T+t), t \in \overline{\mathbb{R}}_+, \tag{6.4}$$

with the operator A, generating an analytic compact C_0-semigroup, where the function f is smooth enough and $f(t,x) = f(t+T,x)$ for any $x \in E$ and $t \in \overline{\mathbb{R}}_+$. Let $u(\cdot;u^0)$ be a solution of the Cauchy problem (6.1) with the initial data $u(0;u^0) = u^0$. This function $u(\cdot;u^0)$ is also a mild solution, i.e., it satisfies the integral equation

$$u(t) = \exp(tA)u^0 + \int_0^t \exp\big((t-s)A\big) f\big(s, u(s)\big) ds, \ t \in \overline{\mathbb{R}}_+. \tag{6.5}$$

Then the shift operator $\mathcal{K}(u^0) = u(T;u^0)$ can be defined, and it maps E into E. If $u(\cdot;x^*)$ is a periodic solution of (6.1), then x^* is a zero of the compact vector field defined by $I - \mathcal{K}$, i.e., $\mathcal{K}(x^*) = x^*$.

Remark 6.3. We assume here that the operator $\big(I - \exp(TA)\big)^{-1}$ exists and is bounded. Meanwhile, it is just enough to assume that $\big(I - \exp(tA)\big)^{-1} \in B(E)$ holds for $t \geq t_0$ with some $t_0 > 0$. This assumption is not restrictive, since, without loss of generality, we can change A by $A - \omega I$ and obtain $\|\exp\big(t(A-\omega)\big)\| \leq Me^{-\delta t}$ for $\delta > 0, t \geq 0$. It follows [29] that $\big(I - \exp(tA)\big)^{-1} \in B(E)$ for any $t > 0$.

Remark 6.4. We say that function f is smooth enough in the sense that it is at least continuous in both arguments, $\sup\limits_{t\in[0,T], \|x\|\leq c_1} \|f(t,x)\| \leq C_2$ and such that there exists the global mild solution of the problem $u'(t) = Au(t) + f\big(t, u(t)\big)$, $u(0) = u^0$, $t \in \overline{\mathbb{R}}_+$.

Definition 6.1. The solution $u(\cdot)$ of the Cauchy problem (6.1) is said to be *stable in the Lyapunov sense* if for any $\epsilon > 0$ there is $\delta > 0$ such that the inequality $\|u(0) - \tilde{u}(0)\| \leq \delta$ implies $\max\limits_{0\leq t<\infty} \|u(t) - \tilde{u}(t)\| \leq \epsilon$, where $\tilde{u}(\cdot)$ is a mild solution of (6.1) with the initial value $\tilde{u}(0)$.

Definition 6.2. The solution $u(\cdot)$ of the Cauchy problem (6.1) is said to be *uniformly asymptotically stable* at the point $u(0)$ if it is stable in the Lyapunov sense, and for any mild solution $\tilde{u}(\cdot)$ of (6.1) with $\|u(0) - \tilde{u}(0)\| \leq \delta$, it follows that $\lim_{t\to\infty} \|u(t) - \tilde{u}(t)\| = 0$ uniformly in $\tilde{u}(\cdot) \in B(u(0);\delta)$, i.e., there is a function $\phi_{u(0),\delta}(\cdot)$ such that $\|u\big(t;u(0)\big) - u\big(t;\tilde{u}(0)\big)\| \leq \phi_{u(0),\delta}(t)$ with $\phi_{u(0),\delta}(t) \to 0$ as $t \to \infty$ and $\|u(0) - \tilde{u}(0)\| \leq \delta$.

Constructive conditions on the operator A and f ensuring that the equation $u'(t) = Au(t) + f(u(t))$, $u(0) = u^0$ is asymptotically k-dimensional are given in [172, 173]. They concern with the location of eigenvalues of A, i.e., $\lambda_{k+1} - \lambda_k > 2L, \lambda_{k+1} > L$.

Theorem 6.3 ([38]). *Assume that conditions* (A) *and* (B'') *hold and compact resolvents* $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ *converge:* $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ *compactly for some* $\lambda \in \rho(A)$. *Assume that*

(i) *the functions* f *and* f_n *are sufficiently smooth, so that there exists an isolated mild solution* $v^*(\cdot)$ *of the periodic problem* (6.4) *with* $v^*(0) = x^*$ *such that the Cauchy problem* (6.1) *with* $u(0) = x^*$ *has a uniformly asymptotically stable isolated solution at the point* x^* *(in this case,* $\operatorname{ind} v^*(\cdot) = 1$);

(ii) $f_n(t, x_n) \to f(t, x)$ *uniformly with respect to* $t \in [0, T]$ *as* $x_n \to x$;

(iii) *the space* E *is separable.*

Then, for almost all n, *the problems*

$$v_n'(t) = A_n v_n(t) + f_n\big(t, v_n(t)\big), v_n(t) = v_n(t+T), t \in \overline{\mathbb{R}}_+, \tag{6.6}$$

have periodic mild solutions $v_n^*(t), t \in [0, T]$, *in the neighborhood of* $p_n v^*(\cdot)$, *where* $v^*(\cdot)$ *is a mild periodic solution of* (6.4) *with* $v^*(0) = x^*$. *Each sequence* $\{v_n^*(\cdot)\}$ *is* $\mathcal{P}$*-compact and* $v_n^*(t) \to v^*(t)$ *uniformly with respect to* $t \in [0, T]$.

We say that a fixed point x^* of the operator $\mathcal{K}$ in Banach lattice E is stable from the above [98] if given $\epsilon > 0$, there is $\delta > 0$ such that $\|\mathcal{K}^k x - x^*\| \le \epsilon$ for all $k \in \mathbb{N}$ if $x^* \preceq x$ and $\|x - x^*\| \le \delta$. Using this notion, we can reformulate Theorem 6.3 for positive semigroups due to the result from [62].

Theorem 6.4. *Let the operators* A_n *and* A *from the problems* (6.4) *and* (6.6) *be compatible and let* E *and* E_n *be order-one spaces and* $e_n \in D(A_n) \cap \operatorname{int} E_n^+$. *Assume that the operators* A_n *have the* POD *property and* $A_n e_n \preceq 0$ *for sufficiently large* n *and compact resolvents* $(\lambda I - A)^{-1}, (\lambda I_n - A_n)^{-1}$ *converge* $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ *compactly for some* $\lambda \in \rho(A)$. *Assume that*

(i) *the functions* f *and* f_n *are sufficiently smooth, bounded and positive, so that there exists a mild solution* $u^*(\cdot)$ *of the Cauchy problem* (6.3) *such that the element* $u^*(0) = x^*$ *is a stable from above and fixed points of operator* $\mathcal{K}$ *with* $x^* \prec y, \mathcal{K}y \preceq y$ *(in this situation* $\operatorname{ind} x^* = 1$);

(ii) $f_n(t, x_n) \to f(t, x)$ *uniformly with respect to* $t \in [0, T]$ *as* $x_n \to x$;

(iii) *the space* E *is separable.*

Then for almost all n, *problems* (6.6) *have periodic mild solutions* $v_n^*(t), t \in [0, T]$ *in the neighborhood of* $p_n v^*(\cdot)$, *where* $v^*(\cdot)$ *is any mild periodic solution of* (6.4) *stable from above. Each sequence* $\{v_n^*(\cdot)\}$ *is* $\mathcal{P}$*-compact and* $v_n^*(t) \to v^*(t)$ *uniformly with respect to* $t \in [0, T]$.

Remark 6.5. The technique which is used here can be applied to the case of condensing operators [3]. For example, the resolvent of Δ in $L^2(\mathbb{R}^d)$ is condensing, but it is not compact.

In [120], the qualitative behavior of spatially semidiscrete finite-element solutions of a semilinear parabolic problem near an unstable hyperbolic equilibrium was studied.

The shadowing approach to the study of the long-time behavior of numerical approximations of semilinear parabolic equations was studied in [119].

Many results contained in this survey can be reformulated for the second-order equation $u'' = Au(t)$ with the operator A generating a C_0-cosine operator function.

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