Journal of Mathematical Sciences, Vol. xxx, No. y, 2003

DIFFERENTIAL EQUATIONS IN BANACH SPACES II. THEORY OF COSINE OPERATOR FUNCTIONS

V. V. Vasil'ev and S. I. Piskarev

UDC 517.986.7; 517.983.6

Dedicated to Vasil'eva Aleksandra Vladimirovna and Piskareva Lidiya Ivanovna, our mothers.

INTRODUCTION

More than 13 years have passed since the fundamental survey [16] was prepared, which, as the author intended, should be the first part of a large work devoted to abstract differential equations and methods for solving them. However, the troubles being in the Russian science during the whole this period have influenced also on the authors, and instead of two years supposed, the preparation of the second part has occupied considerably more time.

During the last 10 years, the work in the field of differential equations in abstract spaces was very active (in foreign countries), and every year several books and a heavy number of papers devoted to this direction appear in the world (of course, the most of them are not available for the Russian reader). At the same time, only two books [33, 75] of such a type appeared being translated by the authors of the present survey and [20], which were edited by Yu. A. Daletskii. Therefore, the work whose second part is proposed to the reader will be undoubtedly useful for the Russian reader. Its style coincides with that of [16], i.e., the material is often presented without proofs, and the main attention is paid to the structure of presentation, although we present certain proofs from foreign sources that are almost inaccessible for Russian readers. From our viewpoint, this allows us to demonstrate clearly the philosophy, to describe the results obtained, and to indicate the main directions of the development of the theory in the framework of a limited volume of the survey.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 113, Functional Analysis, 2002.

Moreover, the authors have prepared a separate edition of the bibliographical index [18], which can serve as a sufficiently complete source of information about the theory of differential equations in abstract spaces during the recent years.

The main object of the study in this part are second-order differential equations that are presented very little in Russian literature up to now. Here, we can only mention the paper [20] written in accordance with the own interests of the author, which does not pretend to the exhausting description of all aspects of the theory. Moreover, the material of the present survey includes the presentation of the abstract Cauchy problem for first- and second-order equations that is not considered in [17].

As was already mentioned in [16], the philosophy of the theory of C_0-cosine operator functions is very close to the operator semigroup theory and often is developed in parallel to it. Therefore, the reader easily draws analogies between the material presented here and that presented in [16]. At the same time, the theory of C_0-cosine operator functions considerably differs from the operator C_0-semigroup theory. First of all, these distinctions concern with the properties inherent to the corresponding parabolic and hyperbolic partial differential equations.

We now present the main notation, a certain part of which was already introduced in [17] and which is also used here.

The set of natural numbers is denoted by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the set of integers by $\mathbb{Z}$, the set of reals by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$. A tuple of numbers $1, 2, ..., m$, $m \in \mathbb{N}$, is denoted by $\overline{1, m}$, the real semiaxis $(0, \infty)$ by $\mathbb{R}_+$, and $[0, \infty)$ by $\overline{\mathbb{R}_+}$.

We denote by E a Banach space over the field of complex numbers with the norm $\|\cdot\|$. For a Hilbert space with the inner product $(\cdot, \cdot)$, we use the symbol H.

The boundary of a set Ω is denoted by $\partial\Omega$, the interior of the set Ω by $\mathrm{int}(\Omega)$, the closure in the strong topology by $\overline{\Omega}$, and, for example, the closure in the weak topology by w-cl-(Ω).

As usual, the space dual to E is denoted by E^*, with elements x^*, y^*, ..., and the value of a functional $x^* \in E^*$ at an element $x \in E$ is written as $\langle x, x^* \rangle$.

The domain and range of an operator A will be written as $D(A)$ and $\mathcal{R}(A)$, respectively, and the null-space (kernel) as $\mathcal{N}(A)$. The set of linear operators acting from $D(A) \subseteq E$ into E is denoted by $L(E)$, and the set of linear continuous operators by $B(E)$. Closed linear operators with dense domain ($\overline{D(A)} = E$) in E are distinguished into the set $\mathcal{C}(E) \subset L(E)$. In the case where operators act from one space E into another F, we write $L(E, F)$ and $B(E, F)$, respectively. The linear variety $D(A)$ endowed with the norm $\|x\|_A := \|x\| + \|Ax\|$ in the case of a closed linear operator becomes a Banach space; we denote it by $\mathcal{D}(A)$.

We use the traditional notation for the resolvent set $\rho(A)$ and spectrum $\sigma(A)$ of an operator A; as usual, the latter is divided into the point spectrum $P\sigma(A)$, the continuous spectrum $C\sigma(A)$, and the residual spectrum $R\sigma(A)$.

Sections 2.2, 2.4, 6.1, 6.3, 7.1–7.2, 9.2, 10.2, 10.3, 12.1–12.4, 12.6–12.10, 13.3–13.6 and Chapter 14 were written by S. I. Piskarev, the other part of the text was prepared by the authors in collaboration.

Chapter 1

CAUCHY PROBLEM AND RESOLVING FAMILIES

Before considering the theory of C_0-cosine operator functions, we describe the general picture of the statement of the well-posed Cauchy problem in a Banach space. As is easily noted, a natural generalization of the concept of solution leads to more general families: integrated semigroups and C-semigroups. These families will be considered in a forthcoming survey in more detail.

1.1. Cauchy Problem for a Complete Differential Equation

Let E be a Banach space, and let $A_0, A_1, ..., A_{m-1}$ be closed linear operators on E, i.e., $A_k \in \mathcal{C}(E), k \in \overline{0, m-1}$. In the Banach space E, let us consider the following abstract Cauchy problem of order m:

$$\begin{cases} u^{(m)}(t) = \sum\limits_{k=0}^{m-1} A_k u^{(k)}(t), & t \in \overline{\mathbb{R}}_+, \\ u^{(k)}(0) = u_k^0, & k \in \overline{0, m-1},\ m \geq 2. \end{cases} \tag{1.1}$$

Definition 1.1.1. A function $u(\cdot) \in C^m(\overline{\mathbb{R}}_+; E)$ is called a *classical solution of problem* (1.1) if $u^{(k)}(t) \in D(A_k)$, $A_k u^{(k)}(\cdot) \in C(\overline{\mathbb{R}}_+; E)$ for $t \in \overline{\mathbb{R}}_+$, $k \in \overline{0, m-1}$, and relations (1.1) hold.

As in [17], we define the propagators $\mathcal{P}_j(t)$, $j = \overline{0, m-1}$, which give a solution of the Cauchy problem (1.1) with initial conditions $u_j^{(k)}(0) = \delta_{jk} u_j^0$ (δ_{jk} is the Kronecker symbol), i.e., $u_j(t) = \mathcal{P}_j(t) u_j^0$.

Definition 1.1.2. The Cauchy problem (1.1) is said to be *uniformly well-posed* if $\mathcal{P}_k(\cdot)x \in C^k(\overline{\mathbb{R}}_+; E), \mathcal{P}_{m-1}^{(k-1)}(t)x \in D(A_k), t \in \overline{\mathbb{R}}_+$, and $A_k \mathcal{P}_{m-1}^{(k-1)} x \in C(\overline{\mathbb{R}}_+; E)$ for any $x \in E$ and $k \in \overline{0, m-1}$.

In the general case, in the Banach space E, problem (1.1) has been studied incompletely. In particular cases, some concepts are introduced, which will be considered in the next chapters in more detail.

Definition 1.1.3. We say that an operator A *generates α times integrated semigroup* with $\alpha \geq 0$ if $(\omega, \infty) \subseteq \rho(A)$ for a certain $\omega \in \mathbb{R}$ and there exists a strongly continuous function $S(\cdot) : [0, \infty) \to B(E)$ such that $\|S(t)\| \leq M e^{\omega t}$, $t \in \overline{\mathbb{R}_+}$, with a certain constant $M \geq 0$ and

$$(\lambda I - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)\, dt$$

for all $\lambda > \max\{\omega, 0\}$. The family $S(\cdot)$ itself is called an *α times integrated semigroup.*

Theorem 1.1.1 ([118]). *Let $m \geq 2$, and let A_{m-1} generate an r times integrated semigroup. Assume that $D(A_i) \supseteq D(A_{m-1})$ for all $i \in \overline{0, m-2}$, and, moreover, $A_i D(A_{m-1}) \subseteq D(A_{m-1}^{i-m+r+2})$ for $i \geq m - r - 1$. Then problem* (1.1) *has a unique exponentially bounded solution for $u_{m-1}^0 \in D(A_{m-1}^{r+1})$, $u_k^0 \in \bigcap\limits_{j=0}^{m-1} D(A_{m-1}^r A_i)$, $k \in \overline{0, m-2}$, and for certain constants $c, \omega > 0$, we have the following estimate for $t \in \overline{\mathbb{R}}_+$:*

$$\|u(t)\| + \|A_{m-1}u(t)\| \leq ce^{\omega t}\left\{\sum_{l=0}^{r+1}\|A_{m-1}^l u_{n-1}^0\| + \sum_{k=0}^{m-2}\sum_{i=0}^{m-2}\sum_{l=0}^{r}\|A_{m-1}^l A_i u_k^0\| + \sum_{k=0}^{m-2}(\|u_k^0\| + \|A_{m-1}u_k^0\|)\right\}.$$

In [193], this theorem was slightly changed by extending the set of initial data and by the absence of the exponential boundedness.

Denote $P(\lambda) := \sum\limits_{i=0}^{m} \lambda^i A_i$ with the domain $D(P) := \bigcap\limits_{i=0}^{m-1} D(A_i)$.

Theorem 1.1.2 ([194]). *The propagators $\mathcal{P}_k^{(k)}(\cdot)$, $k \in \overline{0, m-1}$, are norm-continuous for $t \in \mathbb{R}_+$ iff there exists $\tau_0 \in \mathbb{R}_+$ such that*

$$\lim_{|\omega|\to\infty} \|(\tau_0 + i\omega)^{m-1} P^{-1}(\tau_0 + i\omega)\| = 0, \tag{1.2}$$

$$\lim_{|\omega|\to\infty} \|(\tau_0 + i\omega)^{k-1} \overline{P^{-1}(\tau_0 + i\omega)A_k}\| = 0, \quad k \in \overline{0, m-1}. \tag{1.3}$$

Corollary 1.1.1. *Let conditions* (1.2)–(1.3) *hold. Then for each $k \in \overline{0, m-1}$, the operator $\mathcal{P}_k(t)$ is norm-continuous for $t \in \mathbb{R}_+$.*

The case of time-dependent $A_k = A_k(t)$, $t \in \overline{\mathbb{R}_+}$, was considered, e.g., in [227, 228].

In [295], the conditions for the existence of a unique entire solution of problem (1.1) were presented.

Consider problem (1.1) with $A_k \in \mathcal{C}(E)$ for all $k \in \overline{0, m-1}$; let $D(A_0) \subseteq D(A_k)$ for $k \in \overline{1, m-1}$.

Theorem 1.1.3 ([226]). *Under the assumptions described above, the following conditions are equivalent for the Cauchy problem* (1.1):

(i) *the operator A_0 generates a C_0-semigroup;*

(ii) *for any $u^0, u^1, ..., u^{m-1} \in D(A_0)$, the Cauchy problem* (1.1) *has a unique solution $u(\cdot) \in C^{(m-1)}(\mathbb{R}_+, \mathcal{D}(A_0))$.*

The following theorem on the uniform stability of problem (1.1) holds.

Theorem 1.1.4. *Let an operator A_0 generate a C_0-semigroup, and let $u(\cdot)$ be a solution of the Cauchy problem* (1.1) *with initial conditions u_l^k $(k \in \overline{1, m-1};\ l \to \infty)$, $u_l^k \in D(A)$ for $k \in \overline{1, m-1}$, $u_l^k \to 0$ in E. Then $u_l(\cdot) \to 0$ uniformly on any compact set.*

Theorem 1.1.5 ([226])**.** *Let A_0 generate a C_0-semigroup, $A_k \in \mathcal{C}(E)$, $D(A) \subset D(A_k)$, and let ω be such that for $\operatorname{Re}\lambda > \omega$, there exists a generalized resolvent (pencil resolvent)*

$$R_\lambda := R(\lambda; A_0, ..., A_{m-1}) = (\lambda^m I - \lambda^{m-1} A_{m-1} - ... - \lambda A_1 - A_0)^{-1}$$

(such an ω always exists!). Also, on $D(A)$, let the relation $A_k R_\lambda = R_\lambda A_k$ $(k \in \overline{1, m-1}; \operatorname{Re}\lambda > \omega)$ hold. Then problem (1.1) *is uniformly well-posed, and its solution has the form*

$$u(t) = \sum_{k=0}^{m-1} Q_{m-1-k, m-1}(t) u^k, \tag{1.4}$$

where the operator-valued functions $Q_{m-1-k,m-1}(t)$ are strongly continuous families composing the operator semigroup

$$G(t) = \begin{bmatrix} Q_{0,0}(t) & \dots & Q_{0,m-1}(t) \\ \vdots & \ddots & \vdots \\ Q_{m,0}(t) & \dots & Q_{m-1,m-1}(t) \end{bmatrix}, \quad t \in \mathbb{R}_+,$$

with the generator

$$\Gamma = \begin{bmatrix} A_0 & I & 0 & \dots & 0 \\ A_1 & 0 & I & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & I \\ A_{m-1} & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now let us consider the Cauchy problem for the following equation of order m having the special form:

$$\prod_{j=1}^{m} \left(\frac{d}{dt} - A_j \right) u(t) \equiv \left(\frac{d}{dt} - A_m \right) ... \left(\frac{d}{dt} - A_1 \right) u(t) = 0 \tag{1.5}$$

with initial conditions

$$u^{(k)}(0) = u_k^0, \quad k \in \overline{0, m-1},\ m \geq 2,$$

and operators $A_j \in \mathcal{C}(E), j \in \overline{1, m}$.

Definition 1.1.4. The Cauchy problem (1.5) is said to be *uniformly well-posed* if the following conditions hold:

(i) there exists a solution of the Cauchy problem (1.5) for $u^0, ..., u^{m-1}$ taken from a certain dense set D in E;

(ii) for $u^0, ..., u^{m-1} \in D$, the solution of the Cauchy problem (1.5) has the property

$$\prod_{j=1}^{k} \left(\frac{d}{dt} - A_j \right) u(t) \in C^{m-k}(\overline{\mathbb{R}_+}, E) \tag{1.6}$$

for $k \in \overline{1, m-1}$;

(iii) the uniform stability of the solution of (1.5) is complemented by the following condition on any compact set: the convergence

$$\prod_{j=1}^{k} \left(\frac{d}{dt} - A_j \right) u_p(0) \to 0$$

implies the convergence

$$\prod_{j=1}^{k} \left(\frac{d}{dt} - A_j \right) u_p(t) \to 0$$

uniformly on each compact set in $\overline{\mathbb{R}_+}$ (here, $k \in \overline{1, m-1}$; $p \to \infty$).

Theorem 1.1.6 ([20]). *In the Cauchy problem* (1.5), *let* $A_j \in \mathcal{C}(E)$ $(j \in \overline{1, m})$, *let the intersection of the resolvent sets* $\rho(A_j)$ *of the operators* A_j *be nonempty, and let the set*

$$\tilde{D} \equiv \cap \{ D(A_{i_1} ... A_{i_m}) : i_k \in \overline{1, m} \} \tag{1.7}$$

be dense in E. *Then problem* (1.5) *is uniformly well-posed iff* A_j *generates a* C_0*-semigroup for each* $j \in \overline{1, m}$.

Theorem 1.1.7 ([20]). *Under the conditions of Theorem* 1.1.6, *let the operators* A_j *generate* C_0*-semigroups for* $j \in \overline{1, m}$, *and, moreover, let these semigroups commute*:

$$\exp(tA_i) \exp(sA_j) = \exp(sA_j) \exp(tA_i), \quad t, s \in \mathbb{R}, \quad i, j \in \overline{1, m}. \tag{1.8}$$

Then problem (1.5) *is well posed.*

Theorem 1.1.8. *For the Cauchy problem* (1.5), *let the conditions of Theorem* 1.1.7 *hold, and let* $0 \in \rho(A_i - A_j)$ *for all* $i \neq j$. *Then the condition* $w(t) \in \mathcal{N}\Big(\prod\limits_{i=1}^{m} \Big(\frac{d}{dt} - A_i \Big) \Big)$ *for* $t \in \overline{\mathbb{R}}$ *implies the relation* $w(t) = \sum\limits_{i=1}^{m} w_i(t)$, *where* $w_i(t) \in \mathcal{N}\Big(\frac{d}{dt} - A_i \Big)$, $t \in \mathbb{R}$.

Condition (1.8) in Theorem 1.1.7 can be replaced by a number of conditions imposed on the domains $\mathcal{R}(A_i - A_j)$ for $i \neq j$.

Let $1, \varphi^1, \varphi^2, ..., \varphi^{p-1}$ be roots of pth degree of the unity, i.e., $\varphi^k = e^{\frac{2\pi k}{p} i}$.

Definition 1.1.5. A C_0*-function of the Mittag-Leffler type with a parameter* p is a function $\mathfrak{M} : \mathbb{C} \to B(E)$ having the following properties:

(i) $\sum\limits_{k,l=0}^{p-1} \mathfrak{M}(\varphi^k t + \varphi^l s) = p^2 \mathfrak{M}(t)\mathfrak{M}(s)$ for any $t, s \in \mathbb{R}$;

(ii) $\mathfrak{M}(0) = I$;

(iii) the family of operators $T(t) \equiv \mathfrak{M}(\varphi^k t + \varphi^l s)$, $k, l \in \overline{0, p-1}$, with a fixed $s \in \mathbb{R}$ is strongly continuous in $t \in \mathbb{R}$.

For the Mittag-Leffler C_0-function with a parameter p, the p-generator A is defined by the relation

$$Ax = s\text{-}\lim_{t \to 0} p! \frac{\mathfrak{M}(t) - I}{t^p} x$$

for those x at which the limit exists. It is known that the generator of the Mittag-Leffler C_0-function with a parameter p is a linear closed densely defined operator, and the following relation holds for any $x \in D(A)$:

$$\frac{d^p}{dt^p} \mathfrak{M}(t)x = A\mathfrak{M}(t)x = \mathfrak{M}(t)Ax,$$

and, moreover, $\mathfrak{M}^{(k)}(0) = 0$ for $k \in \overline{0, p-1}$.

For the Mittag-Leffler C_0-function with a parameter p, the perturbation theorems of the Philips–Miyadera type hold (see [17]).

Theorem 1.1.9 ([32]). *Let* A *generate a Mittag-Leffler* C_0*-function with a parameter* p*, and let* $\|\mathfrak{M}(t)\| \leq Me^{\omega t}$, $t \in \mathbb{R}$. *Then for any* $B \in B(E)$*, the operator* $A + B$ *generates a Mittag-Leffler* C_0*-function with the parameter* p.

Proposition 1.1.1. *The Mittag-Leffler* C_0*-function with a parameter* p *is a* C_0*-group of operators in the case of* $p = 1$*, and in the case of* $p = 2$*, it is a* C_0*-cosine operator-valued function. A Mittag-Leffler* C_0*-function with a parameter* p *has a bounded generator* $A \in B(E)$ *for* $p \geq 3$.

In the simplest case $m = 2$, for example, the following theorems hold for problem (1.1).

Theorem 1.1.10 ([226]). *Let the Cauchy problem* (1.1) *be uniformly well-posed for* $m = 2$*, and let* $\mathcal{P}_1'(t)E \in \mathcal{D}(A_1)$ *for* $t \in \overline{\mathbb{R}}_+$. *Then* A_0 *generates a* C_0*-cosine operator-valued function on* E.

Theorem 1.1.11 ([226]). *Let* $A_1 \in B(E)$. *Then the Cauchy problem* (1.1) *is uniformly well posed for* $m = 2$ *iff* A_0 *generates a* C_0*-cosine operator-valued function on* E.

However, in the general case, even for $m = 2$, the Cauchy problem (1.1) turns out to be very complicated. First, in [135], H. O. Fattorini has presented an example of the Cauchy problem (1.1) that has a solution for $m = 2$, but this solution is not exponentially bounded. Second, in contrast to the

Cauchy problem for $m = 1$, the case $m = 2$ admits more flexibility in the sense of well-posedness of its statement.

As one of the variants, let us present the approach coming back to H. O. Fattorini. The constructions used in proving these theorems practically completely repeat the techniques used in proving the assertions concerning C_0-cosine and C_0-sine operator-valued functions (see also [30]).

Consider the Cauchy problem

$$u''(t) + Bu'(t) + Au(t) = 0, \quad t \in \overline{\mathbb{R}}_+, \quad u(0) = u^0, \quad u'(0) = u^1, \tag{1.9}$$

with $A, B \in \mathcal{C}(E)$.

Definition 1.1.6. We say that the operators A and B generate *M, N-families of operators on E* if the following conditions hold:

(i) $M(t)$ and $BN(t)$ are strongly continuous in $t \in \overline{\mathbb{R}}_+$, and the function $N(t)x$ is strongly continuously differentiable in $t \in \overline{\mathbb{R}}_+$ for any $x \in E$;

(ii) the set $\hat{E} = \{x \in E : M(t)x$ is strongly differentiable in $t \in \overline{\mathbb{R}}_+$, and $BM(t)x$ is continuous in $t \in \overline{\mathbb{R}}_+\}$ is dense in E;

(iii) the operator $A = -M''(0)$ is B-closed, and $Bx = -N''(0)x$ for all $x \in \hat{E}$;

(iv) $M(0) = N'(0) = I$ and $N(0) = 0$;

(v) $M(t+s)x = M(t)M(s)x + N(t)M'(s)x$ for all $x \in \hat{E}$ and $t, s \in \overline{\mathbb{R}}_+$;

(vi) $N(t+s) = M(t)N(s) + N(t)N'(s)$ for all $t, s \in \overline{\mathbb{R}}_+$.

Theorem 1.1.12 ([30])**.** *Let A and B generate M, N-families. Then*

(i) *A is closed, $D(A) \cap D(B) \subseteq \hat{E} \subseteq D(B)$, and $D(A) \cap D(B)$ is dense in E;*

(ii) *the families M and N are uniquely defined by the operators A and B;*

(iii) *$M'(0)x = 0$ for all $x \in D(M'(0))$;*

(iv) *$M'(t)x = -N(t)Ax$ for all $x \in D(A)$ and $t \in \overline{\mathbb{R}}_+$;*

(v) *$N'(t)x = M(t)x - N(t)Bx$ for all $x \in \hat{E}$ and $t \in \overline{\mathbb{R}}_+$;*

(vi) *$N''(t)x + N'(t)Bx + N(t)Ax = 0$ for all $x \in D(A) \cap D(B)$ and $t \in \overline{\mathbb{R}}_+$;*

(vii) *for all $x \in \hat{E}$ and $t \in \overline{\mathbb{R}}_+$, the element $N'(t)x \in \hat{E}$, $N(t)x \in D(A)$, and $N''(t)x - x + BN'(t)x + AN(t)x = 0$;*

(viii) *for all $x \in E$ and $t \in \overline{\mathbb{R}}_+$, the element $\int\limits_0^t N(s)x\,ds \in D(A)$ and $N''(t)x - x + BN(t)x + A\int_0^t N(s)x\,ds = 0$;*

(ix) *for all $x \in E$ and $t \in \overline{\mathbb{R}}_+$, the element $\int\limits_0^t N(s)x\,ds \in \hat{E}$, $M(t)x \in D(A)$ and $M''(t)x + BM'(t)x + AM(t)x = 0$;*

(x) *for all* $x \in D(A) \bigcup \hat{E}$ *and* $t \in \overline{\mathbb{R}}_+$, *the element* $M(t)x - x \in \hat{E}$, $\int\limits_0^t M(s)x\,ds \in D(A)$, *and* $M'x + B(M - I)x + A\int\limits_0^t M(s)xds = 0$;

(xi) *there exist constants* $C, \omega \geq 0$ *such that*

$$\|M(t)\|, \|N(t)\|, \|BN(t)\|, \|N'(t)\| \leq Ce^{\omega t}, \quad t \in \overline{\mathbb{R}}_+,$$

and for all $x \in \hat{E}$, *there exist constants* $C, \omega \geq 0$ *such that*

$$\|M'(t)x\|, \|BM(t)x\| \leq C(x)e^{\omega t}, \quad t \in \overline{\mathbb{R}}_+;$$

(xii) *the operator* $\lambda^2 + \lambda B + A$ *is closable for all* $\lambda \in \mathbb{C}$;

(xiii) *there exists a constant* $\omega \in \overline{\mathbb{R}}_+$ *such that* $\lambda \in \rho(A, B)$ *for all* λ *with* $\operatorname{Re}\lambda > \omega$ *and*

$$\Delta(\lambda)x := (\lambda^2 I + \lambda B + A)^{-1}x = \int_0^\infty e^{-\lambda t}N(t)x\,dt \quad \text{for} \quad x \in E;$$

$$\Delta(\lambda)(I + B)x = \int_0^\infty e^{-\lambda t}M(t)xdt \quad \text{for} \quad x \in \hat{E};$$

(xiv) $\lambda^2\Delta(\lambda)x \to x$ *as* $\lambda \to \infty$ *for all* $x \in E$.

The following analog of Theorem 2.1.1 from [17] holds.

Theorem 1.1.13 ([294])**.** *Operators* A *and* B *generate* M, N*-families iff the following conditions hold*:

(i) *the operators* A *and* B *are closed, and* $D(A) \bigcap D(B)$ *is dense in* E;

(ii) *there exist constant* $C, \omega \geq 0$ *such that* $\lambda \in \rho(A, B)$, *and for* $\operatorname{Re}\lambda > \omega$, *the operator* $\Delta(\lambda)A$ *is closable and*

$$\|(\lambda\Delta(\lambda))^{(k)}\|, \|(B\Delta(\lambda))^{(k)}\|, \|(\overline{\Delta(\lambda)B})^{(k)}\| \leq \frac{Ck!}{(\operatorname{Re}\lambda - \omega)^{k+1}} \quad \text{for} \quad k \in \mathbb{N}, \quad \operatorname{Re}\lambda > \omega, \tag{1.10}$$

where $\overline{\Delta(\lambda)B}$ *is a bounded extension of the operator* $\Delta(\lambda)B$ *with the domain* $D(A) \cap D(B)$ *and* $(\cdot)^{(k)}$ *is the derivative of order* k *in* λ.

In the case where A and B commute, instead of the estimate with the operator B in (1.10), it can be, e.g.,

$$\left\| \frac{d^k}{d\lambda^k}\Big((\lambda I - A)\Delta(\lambda)\Big) \right\| \leq \frac{Mk!}{(\operatorname{Re}\lambda - \omega)^{k+1}}, \; k \in \mathbb{N}_0$$

(see [30]).

If $A = 0$, then A and B generate M, N-families iff B generates a C_0-semigroup.

Let $D(B) \subseteq D(A)$, and let $\rho(B) \neq \varnothing$. If $(\lambda_0 I - B)^{-1}A$ has a bounded extension for a certain point $\lambda_0 \in \rho(B)$, then A and B generate M, N-families iff B generates a C_0-semigroup.

Proposition 1.1.2 ([294]). *Let B be dominated by A with exponent $0 \le \alpha \le 1$, i.e., $D(A) \subseteq D(B)$ and $\|Bx\| \le C_\alpha \|x\|^{1-\alpha}\|Ax\|^\alpha$ for all $x \in D(A)$, and let A and B commute. If $-A$ generates a C_0-cosine operator-valued function and $\|(\lambda^2 I + A)^{-1}\| \le C|\lambda|^{-2}$ for $\operatorname{Re}\lambda > \omega$, then A and B generate M and N families.*

Now let us consider an analytic extension of a solution of Eq. (1.9) to the sector $\Sigma(\theta) = \{z \in \mathbb{C} : z \ne 0, |\arg z| < \theta\}$.

Theorem 1.1.14 ([294]). *For given $\theta, \omega \ge 0$, the following conditions are equivalent*:

(i) *the Cauchy problem* (1.9) *is uniformly well posed, the families M, N can be analytically extended to the sector $\Sigma(\theta) = \{z \in \mathbb{C} : z \ne 0, |\arg z| < \theta\}$, for any z we have the embedding $N(z)E \subseteq D(B)$, and $BN(\cdot)$ is analytic in $\Sigma(\theta)$. Moreover, for each $\theta' \in (0,\theta)$, $x \in E$,*

$$\lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} N'(z)x = 0,\ \lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} BN(z)x = 0,\ \lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} M(z)x = x,\ \lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} N(z)x = 0,$$

and there exists a constant $C'_\theta > 0$ such that

$$\|N'(z)\|, \|BN(z)\|, \|M(z)\| \le C'_\theta e^{\omega \operatorname{Re} z} \quad \textit{for all} \quad z \in \Sigma(\theta);$$

(ii) *the set $D(A) \cap D(B)$ is dense in E. For each $\theta' \in (0,\theta)$, there exists $M'_\theta > 0$ such that for*

$$\lambda \in \Sigma(\theta', \omega) = \left\{\lambda \in \mathbb{C} : \lambda \ne \omega, |\arg(\lambda - \omega)| < \frac{\pi}{2} + \theta'\right\},$$

the operator $\Delta(\lambda) := (\lambda^2 + \lambda B + A)^{-1} \in B(E)$ exists, the operator $\Delta(\lambda)A$ is closable, and

$$\|\lambda\Delta(\lambda)\| \le \frac{M}{|\lambda-\omega|}, \quad \|B\Delta(\lambda)\| \le \frac{M}{|\lambda-\omega|}, \quad \|\overline{\Delta(\lambda)B_0}\| \le \frac{M}{|\lambda-\omega|},$$

where $B_0 \subseteq B$ with $D(B_0) = D(A) \cap D(B)$. Moreover, in this case, we have

$$N''(z) + BN'(z) + AN(z) = 0, \quad M''(z) + BM'(z) + AM(z) = 0,$$

where $AN(z)$ and $AM(z)$ are analytic in $\Sigma(\theta)$. For each $\theta' \in (0,\theta)$,

$$\lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} M'(z)x = 0 \quad \textit{for any} \quad x \in D(A).$$

The existence and uniqueness of solutions of Eq. (1.9) under certain "hyperbolic" conditions is considered in [230]. Problem (1.9) in the case of nonlinear B was considered in [188].

In this section, we present certain statements of the Cauchy problem for equations of the first and second orders. Equations of the first order were already considered in the paper [17] but, however, only in connection with C_0-semigroups on the space E.

As was already noted, different statements of the Cauchy problem are possible. We now present certain arguments that show that a solution is given not by C_0-families on the whole space E.

Definition 1.2.1. An *integrated solution* of the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = x, \tag{1.11}$$

is a continuously differentiable function $v(\cdot) : \mathbb{R}_+ \to E$ such that
(i) $v(\cdot) \in C([0,\infty); \mathcal{D}(A))$ and (ii) $\dfrac{dv}{dt}(t) = Av(t) + x$, $v(0) = 0$.

Definition 1.2.2. Denote by $\mathcal{Z}(A)$ the *resolving set* of the operator A, i.e., the set of all $x \in E$ for which the Cauchy problem (1.11) has an integrated solution.

Proposition 1.2.1. *Let $\mathcal{Z}(A)$ be the resolving subspace endowed with the family of seminorms*

$$\|x\|_{a,b} = \sup_{t\in[a,b]} \|u(t)\|, \qquad a, b \in \mathbb{R}_+. \tag{1.12}$$

Then $\mathcal{Z}(A)$ is a Frechét space and $T(t)x = u(t)$ is a locally equicontinuous semigroup generated by the operator $A|_{\mathcal{Z}(A)}$.

Recall the definition of entire vectors, which is equivalent to [17, Definition 3.1.3].

Definition 1.2.3. Denote by $\mathfrak{U}_c(A)$ the set of entire vectors of an operator A, i.e., the set of $x \in D(A^\infty)$ such that for any $t \in \overline{\mathbb{R}}_+$,

$$\sum_{k=0}^{\infty} \|A^k x\| \frac{t^k}{k!} < \infty.$$

Proposition 1.2.2 ([117]). *Any linear closed operator on E with the resolvent set $\rho(A)$ containing the semiaxis (ω,∞) generates a C_0-semigroup on a certain maximal subspace in E.*

Proposition 1.2.3 ([117]). *For any closed linear operator A, there exists a maximal Frechét space $\mathcal{Z}(A)$ such that $\mathcal{Z}(A) \subseteq E$ and the Cauchy problem* (1.11) *is automatically well posed on $\mathcal{Z}(A)$.*

As is seen from this proposition, the requirement of existence of a C_0-semigroup is very restrictive. At the same time, namely for C_0-families of operators, the technical tools for studying approximate methods are most well elaborated.

Theorem 1.2.1 ([117]). *Let $A \in \mathcal{C}(E)$. Then $\mathfrak{U}_c(A) = \{x \in E :$ problem (1.11) has an entire solution$\}$ and $\mathfrak{U}_c(A) \subseteq \mathcal{Z}(A)$.*

Theorem 1.2.2 ([85]). *Let A generate an analytic C_0-semigroup on E. Then $\mathfrak{U}_c(A) = \mathcal{Z}(A)$, and, moreover, the equality holds topologically and algebraically.*

As is known, a self-adjoint operator $A^* = A \leq 0$ on a Hilbert space H generates an analytic C_0-semigroup, as well as a C_0-cosine operator function. Moreover, in this case, by the Stone theorem, the operator iA generates an unitary C_0-group on H. At the same time, the practical problems often require the omitting of the self-adjointness of the operator and Hilbert property of the initial space. Therefore, to reveal whether a concrete operator generates a C_0-semigroup or not is not a simple but often a complicated independent problem. Here, we present examples showing when the verification of generation of C_0-families on the Banach space E is possible.

Theorem 1.2.3 ([43]). *For $A \in \mathcal{C}(E)$ to be a generator of an analytic C_0-semigroup, it is necessary and sufficient that there exist numbers ν, ω, and $\alpha > 1$ such that the following inequality holds for all* $\operatorname{Re}\lambda > \omega_0$:

$$\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M|\lambda|^\nu}{(\operatorname{Re}\lambda)^{\nu+\alpha-1}(\operatorname{Re}\lambda - \omega)};$$

moreover, the following representation holds for this semigroup:

$$\exp(zA) = -\frac{\alpha}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{z\mu^\alpha}\mu^{\alpha-1}(\mu^\alpha I - A)^{-1}\, d\mu$$

for $z \in \left\{z \in \mathbb{C} : \operatorname{Im} z < \operatorname{Re} z \left|\cot \frac{\alpha\pi}{2}\right|\right\}$.

Let $\Omega \subset \mathbb{R}^d$ be a certain domain. Denote by $\mathfrak{C}(\Omega)$ the space of uniformly continuous bounded functions on Ω with the norm

$$\|v(\cdot)\|_{\mathfrak{C}(\Omega)} = \sup_{x\in\Omega}|v(x)|,$$

and let $\mathfrak{C}_\rho(\Omega) = \{v(\cdot) : \rho(\cdot)v(\cdot) \in \mathfrak{C}(\Omega),\ \rho(t) \geq 0\}$, $\|v(\cdot)\|_{\mathfrak{C}_\rho(\Omega)} = \|\rho(\cdot)v(\cdot)\|_{C(\Omega)}$.

Example 1.2.1 ([43]). Let $\Omega = [0,1]$, and let $Av = v''(\cdot)$ with $D(A) = \{v(\cdot) : v \in C(\Omega), Av \in C(\Omega), v'(0) = v'(1) = 0\}$. Then $A \in \mathcal{H}\left(\omega, \frac{\pi}{2}\right)$ in $C([0,1])$.

At the same time, for the operator $A_0 v = v''(\cdot)$ with $D(A_0) = \{v(\cdot) : v \in C([0,1]), v(0) = v(1) = 0\}$, we have $A_0 \in \mathcal{H}(0,\beta)$ on $C_0([0,1]) = \{v(\cdot) : v \in C([0,1]), v(0) = v(1) = 0\}$.

Finally, the operator $A_\rho v = v''$ with $D(A_\rho) = \{v(\cdot) : \rho(\cdot)v(\cdot) \in C([0,1]),\ A_\rho v \in \mathfrak{C}_\rho([0,1])\}$ generates an analytic C_0-semigroup with the estimate

$$\|\exp(tA_\rho)\|_{\mathfrak{C}_\rho([0,1])} \leq e^{-\pi^2 t}, \quad t \in \overline{\mathbb{R}_+}.$$

Example 1.2.2 ([43, 268]). The Laplace operator $\Delta v = \sum_{j=1}^{d} \frac{\partial^2 v(x)}{\partial x_j^2}$, $x \in \mathbb{R}^d$, for $1 < p < \infty$ gives a generator of an analytic C_0-semigroup on $E = W^{2,p}(\mathbb{R}^d)$.

Here, it is appropriate to recall (see [167]) that the operator $i\Delta$ does not generate a C_0-semigroup on $L^p(\mathbb{R}^d)$ for $p \neq 2$. Moreover, the operator Δ generates a C_0-cosine operator function iff $p = 2$ or $d = 1$.

Also, we note (see [126]) that the operator $-(i\Delta)^{1/2}$ does not generate a C_0-semigroup on $L^1(\mathbb{R}^1)$.

Example 1.2.3 ([126]). The Laplace operator Δ on $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$, generates an α times integrated cosine operator function for $\alpha > (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|$.

Example 1.2.4 ([125]). Let $\tilde{A}$ be a strongly elliptic operator on $\Omega \subseteq \mathbb{R}^d$. Denote by $T_r(\cdot)$ the C_0-semigroup generated by the operators $\tilde{A}$ with the Dirichlet or Neumann conditions on the boundary in $L^r(\Omega)$. Then there exists an analytic C_0-semigroup $T_p(\cdot)$ with the angle $\pi/2$ in $L^p(\Omega)$ such that $T_p(t)x = T_r(t)x$ for all $x \in L^p(\Omega) \cap L^r(\Omega)$.

Example 1.2.5 ([126]). Let $1 < p < \infty$, let the operator $\tilde{A}_p$ generate a semigroup $T_p(\cdot)$, and let $\mu(\Omega) < \infty$. Then $\tilde{A}_p$ generates an α times integrated cosine operator function on $L^p(\mathbb{R}^d)$ for

$$\alpha > \frac{d}{2}\left|\frac{1}{2} - \frac{1}{p}\right| + \frac{1}{2}.$$

Example 1.2.6 ([35]). Let $\Omega = \mathbb{R}_+$. The operator $(Av)(x) = v''(x) + \frac{a}{x}v'(x) + \frac{c}{x}v(x)$ generates an analytic C_0-semigroup for $a, c \in \mathbb{R}$ and $D(A) = \{v(\cdot) : v \in \mathfrak{C}(\overline{\mathbb{R}_+}), Av \in C(\overline{\mathbb{R}_+})\}$ iff $c \leq 0$.

Example 1.2.7 ([43]). Let $\Omega = [-1, 1]$. Then the operator $(Av)(x) = \frac{d}{dx}\left((1-x^2)\frac{dv(x)}{dx}\right)$ with $D(A) = \{v(\cdot) : v \in \mathfrak{C}(\Omega),\ Av \in \mathfrak{C}(\Omega)\}$ generates a C_0-semigroup.

Example 1.2.8 ([43]). Let $(Av)(x) = v''(x) + q(x)v(x)$, $x \in \mathbb{R}$. Denote by S_p the Banach space of Stepanov functions, i.e., the space of functions on $\mathbb{R}$ with the norm

$$\|v(\cdot)\|_{p,l} = \sup_{x \in \mathbb{R}} \left(\frac{1}{l}\int_x^{x+l} |f(s)|^p \, ds\right)^{\frac{1}{p}}, \ l > 0, \ p \geq 1.$$

It is known that for different l, the norms are equivalent. For the operator $A \in \mathcal{H}(\omega, \beta)$ on $C(\mathbb{R})$, it suffices, and in the case $q(x) \geq c > -\infty$, it is necessary that $q(\cdot) \in S_1$.

Denote $H_{-1}(z) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2 - 2sz} \, ds$ and

$$H_\tau(t)(\mu^2 I - A)^{-1} = -\frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} H_{-1}(\lambda\tau)e^{\lambda t}\lambda(\lambda^2 I - A)^{-1} \, d\lambda.$$

Theorem 1.2.4 ([35]). *For $A \in \mathcal{C}(M,\omega)$, it is necessary and sufficient that this operator be the generator of an analytic C_0-semigroup, and for each $t \in [0,T]$, the estimate*

$$\|H_\tau(t)(\mu^2 I - A)^{-1}\| \le M(t) \tag{1.13}$$

hold uniformly in $\tau \in (0,\varepsilon)$, $\varepsilon > 0$. In this case,

$$C(t,A) = s\text{-}\lim_{\tau\to 0}(H_\tau(t) + H_\tau(-t))(\mu^2 I - A)^{-1}, \qquad t \in \overline{\mathbb{R}_+}.$$

Example 1.2.9 ([35]). Let the operator A be given as in Example 1.2.6. Then for the function

$$f_\tau(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ \dfrac{x-1}{2\tau} & \text{if } x \in [1, 1+2\tau), \\ 1 & \text{if } x \in [1+2\tau, \infty) \end{cases}$$

we have $\|H_\tau(1)(\lambda^2 I - A)^{-1} f_\tau\| > \dfrac{M}{\tau}$, and, therefore, by Theorem 1.2.4, such an operator A does not generate a C_0-cosine operator function.

Example 1.2.10 ([43]). Consider the operator A from Example 1.2.6 but on the space $\mathfrak{C}_\rho(\overline{\mathbb{R}_+})$ with $\rho(x) = xe^{x\gamma}$, $x \in \overline{\mathbb{R}_+}$, $\gamma \in \mathbb{R}$. Then $\|H_\tau(t)(\lambda^2 I - A)^{-1} v\|_{\mathfrak{C}_\rho} \le Me^{|\mu t|}\|v\|_{\mathfrak{C}_\rho}$, and, therefore, $A \in \mathcal{C}(M,\omega)$.

Example 1.2.11 ([43]). Let A be given as in Example 1.2.8. For $A \in \mathcal{C}(M,\omega)$ on the space $C(\mathbb{R})$ it is sufficient, and in the case $q(x) \ge c > -\infty$, it is necessary that $q(\cdot) \in S_1$.

Consider the problem

$$\frac{\partial^2 u(t,x)}{\partial t^2} = x^m \frac{\partial^2 u(t,x)}{\partial x^2} + \alpha x^{m-1}\frac{\partial u(t,x)}{\partial x} \tag{1.14}$$

where $m > 0$, $x > 0$, and initial conditions $\lim_{t\to 0} u(t,x) = \varphi(x)$, $\lim_{t\to 0}\dfrac{\partial u(t,x)}{\partial t} = \psi(x)$ for any $x \in \mathbb{R}_+$, where $\varphi, \psi \in C^{(2)}(\mathbb{R}_+) \cap E$, and E is the Banach space of functions $\varphi \in C(\mathbb{R}_+)$ such that $\lim_{x\to 0}\varphi(x) = \lim_{x\to\infty}\varphi(x) = 0$ with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}_+}|\varphi(x)|$.

Definition 1.2.4. Problem (1.14) is said to be *uniformly well-posed* if for any compact set $J \subset \mathbb{R}_+$, we have $\max_{t\in J}|u(t,x)| \le M(\|\varphi\| + \|\psi\|)$.

Example 1.2.12 ([43]). For the operator $(Av)(x) = x^m v''(x) + \alpha x^{m-1}v'(x)$ with $D(A) = \{v \in E : v \in C^{(2)}(\mathbb{R}_+) \cap E,\ Av \in E\}$ on the space E just described, condition (1.13) holds for $0 \le \dfrac{2\alpha - (1+\alpha)^m}{2-m} < 1$. For $\dfrac{2\alpha - (1+\alpha)^m}{2-m} \ge 1$, the operator A does not generate a C_0-cosine operator function.

Example 1.2.13 ([43]). For the operator $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ defined on the space $C(\mathbb{R}^d)$, the operator Δ^{2k+1} on $C(\mathbb{R}^d)$ generates a C_0-cosine operator function iff $d \leq 4k+1$.

Here, in connection with Example 1.2.13, it is relevant to note that for any $A \in \mathcal{C}(M,\omega)$, every polynomial $P(A) = A^{2m+1} + \sum_{k=0}^{2m} c_k A^k$, $c_k \in \mathbb{R}$, generates an analytic semigroup.

Moreover, in the case of an even m, the operator $(-1)^{m+1}A^m$ does not necessarily generate a C_0-cosine operator function.

Proposition 1.2.4 ([177]). *Let $A \in \mathcal{C}(M,0)$. Then for any $k \in \mathbb{N}$, the operator $(-1)^k A^{2^k}$ generates an α times integrated cosine operator function for a certain $\alpha > 0$. Moreover, $(-1)^k A^{2^k} \in \mathcal{H}(\omega, \pi/2)$.*

Theorem 1.2.5 ([231]). *Let $A \in \mathcal{H}\left(0, \frac{\pi}{2}\right)$, and let $m \in \mathbb{N}$. Let $B_i \in B(E)$, $i \in \overline{1,m}$. Then the operator $(-1)^{m+1}A^m + B_1 A^{m-1} + \ldots + B_{m-1}A + B_m$ generates an analytic C_0-semigroup with the angle $\frac{\pi}{2}$.*

An analogous assertion is not true for C_0-cosine operator functions!

Theorem 1.2.6 ([178]). *Let $\{A_j\}_{j=1}^m$ be resolvent commuting operators, and let $A_j \in \mathcal{C}(M,0)$, $j \in \overline{1,m}$, be given on E. Define $A_0 = \sum_{j=1}^{m} A_j$, $D(A_0) = \bigcap_{j=1}^{m} D(A_j)$. Then the operator A_0 is closable and $\overline{A_0}$ generates an α times integrated cosine operator function for $\alpha \geq \frac{m-1}{2}$.*

Moreover, this α times integrated semigroup satisfies the estimate $\|S(t)\| \leq M_\alpha t^\alpha$, $t \in \overline{\mathbb{R}_+}$ for certain $M_\alpha > 0$ and $\alpha \geq \frac{m-1}{2}$.

Theorem 1.2.7 ([177]). *Under the conditions and notation of Theorem 1.2.6, the operator $i\overline{A_0}$ generates a β times integrated semigroup for $\beta > m/2$.*

Proposition 1.2.5 ([178]). *Under the conditions of Theorem 1.2.6 and an additional assumption that the space $E = H$ is a Hilbert space, $\overline{A_0}$ generates a C_0-cosine operator function.*

Proposition 1.2.6 ([178]). *Let the conditions of Theorem 1.2.6 hold and, additionally, let $E = H$ be a Hilbert space and a Banach lattice, and, moreover, let $C(t, A_j)H_+ \subseteq H_+$, $t \in \mathbb{R}$, $j \in \overline{1,m}$. Define*

$$C_k(t) = \begin{cases} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C(s, \overline{A_0})\, ds & \text{for } k \geq 1, \\ C(t, \overline{A_0}) & \text{for } k = 0. \end{cases}$$

Then $C_k(\cdot)$ are positive for $k \geq \left[\frac{m}{2}\right]$.

Example 1.2.14 ([162])**.** Let $E = L^p(\mathbb{R}^d)$, $1 < p < \infty$. Then the Laplace operator Δ with $D(\Delta) = W^{2,p}(\mathbb{R}^d)$ generates an α times integrated cosine operator function iff $\alpha \geq (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|$.

In [296], concrete differential operators are studied for revealing whether or not they generate a well-posed Cauchy problem for a complete second-order equation.

1.3. Resolvent Families

For functions $k(\cdot) \in L^p_{loc}(\mathbb{R}_+)$ and $g(\cdot) \in W^{1,1}([0,T];E)$, let us consider the Volterra equation

$$u(t) = g(t) + \int_0^t k(t-s)Au(s)\,ds, \quad t \in [0,T]. \tag{1.15}$$

Definition 1.3.1. A strongly continuous family of bounded linear operators $\{R(t) : t \in \overline{\mathbb{R}}_+\}$ on E is called a *resolvent family* for (1.15) if it commutes with the operator A and

$$R(t)x = x + \int_0^t k(t-s)AR(s)x\,ds \qquad \text{for} \quad x \in D(A),\ t \in \overline{\mathbb{R}}_+.$$

If there exists a resolvent family, then any solution of Eq. (1.15) is represented in the form

$$u(t) = R(t)g(0) + \int_0^t k(t-s)g'(s)\,ds, \quad t \in [0,T]. \tag{1.16}$$

Theorem 1.3.1 ([114, 150, 244])**.** *Let $R(\cdot)$ be a strongly continuous family of operators on $\overline{\mathbb{R}}_+$ such that $\|R(t)\| \leq Me^{\omega t}$ and $|k(t)| \leq Me^{\omega t}$, $t \in \overline{\mathbb{R}}_+$. Then $R(\cdot)$ is a resolvent family iff the following conditions hold*:

(i) $\hat{k}(\lambda) \neq 0$ *and* $\dfrac{1}{\lambda\hat{k}(\lambda)} \in \rho(A)$ *for all* $\lambda \geq \omega$;

(ii)

$$(I - \lambda\hat{k}(\lambda)A)^{-1}x = \int_0^\infty e^{\lambda t}R(t)x\,dt$$

for all $x \in E$ and $\lambda > \omega$, where $\hat{k}(\cdot)$ is the Laplace transform of the function $k(\cdot)$.

In particular, it should be noted that for $k(t) \equiv 1$, the resolvent family is a C_0-semigroup of operators, and for $k(t) = t$, it is a C_0-cosine operator function. Therefore, the proof of a number of assertions on properties related to C_0-semigroups and C_0-cosine operator functions can be obtained from assertions related to resolvent families.

For the kernel $k(\cdot)$ satisfying certain restrictions (positivity and bounded variation) for a resolvent family, many results that hold for C_0-semigroups and C_0-cosine operator functions were reproved. So, for example, in [196–198], C. Lizama has reproved the assertions on the compactness properties, uniform continuity, and periodicity. In [170], Jung Chan Chang and S.-Y. Shaw have reproved the theorems on multiplicative and additive perturbations.

For a second-order equation, let us consider the so-called incomplete Cauchy problems

$$\begin{aligned} &u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ &u(0) = u^0, \quad \sup_{t\in\mathbb{R}_+} \|u(t)\| < \infty; \end{aligned} \tag{1.17}$$

$$\begin{aligned} &u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ &\lim_{t\to 0+} u(t) = u^0, \quad \lim_{t\to\infty} \|u(t)\| = 0; \end{aligned} \tag{1.18}$$

$$\begin{aligned} &u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ &u(0) = u^0, \quad \lim_{t\to\infty} \|u(t)\| = 0. \end{aligned} \tag{1.19}$$

Incomplete Cauchy problems were studied in [117, 137, 140].

Proposition 1.4.1. *An operator A has the square root $\sqrt{A}$ such that $\exp(t\sqrt{A})$ is a bounded analytic C_0-semigroup iff problem (1.17) has a unique solution for each $u^0 \in D(A)$ and this solution is analytically continued to a certain sector containing the semiaxis $\mathbb{R}_+$.*

Proposition 1.4.2. *Let there exist $\sqrt{A}$ generating a differentiable C_0-semigroup such that s-$\lim\limits_{t\to\infty} \exp(t\sqrt{A}) = 0$. Then problem (1.18) has a unique solution for any $u^0 \in E$.*

Proposition 1.4.3. *Let $\sqrt{A}$ generate a C_0-semigroup such that s-$\lim\limits_{t\to\infty} \exp(t\sqrt{A}) = 0$. Then problem (1.19) has a unique solution for each $u^0 \in D(A)$.*

Definition 1.4.1. A C_0-semigroup $\exp(\cdot A)$ is called a *C_0-semigroup stable in degree $q \in \mathbb{N}$* if *s*-$\lim\limits_{t\to\infty} \exp(tA)x = 0$ for each $x \in D(A^q)$. A C_0-semigroup stable in degree 0 is said to be *uniformly stable.*

Theorem 1.4.1. *Assume that an operator $\mathfrak{B}$ generates a C_0-semigroup stable in degree 2 and a function $v(\cdot)$ has a continuous second derivative and satisfies the equation*

$$v''(t) = \mathfrak{B}^2 v(t), \quad t \in \mathbb{R}_+,$$

and, moreover, s-$\lim\limits_{t\to\infty} v(t) = 0$. Then $v(t) = \exp(t\mathfrak{B})v(0)$, $t \in \overline{\mathbb{R}_+}$.

Theorem 1.4.2. *Let $A = \mathfrak{B}^2$, where the operator $\mathfrak{B}$ generates a C_0-semigroup. Then*

(i) *if $\exp(\cdot\mathfrak{B})$ is stable in degree 2, then problem (1.19) is well posed;*

(ii) *if* $\exp(\cdot\mathfrak{B})$ *is stable in degree* 1*, then the following problem is well posed:*

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = x, \\ \lim_{t\to\infty} \|u(t)\| = 0, \quad \lim_{t\to\infty} \|u'(t)\| = 0; \end{gathered} \tag{1.20}$$

(iii) *if* $\exp(\cdot\mathfrak{B})$ *is uniformly stable, then the following problem is well posed:*

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = x, \\ \lim_{t\to\infty} \|u^{(k)}(t)\| = 0, \quad k \in \mathbb{N}_0. \end{gathered} \tag{1.21}$$

Proposition 1.4.4. *Let* $\rho(A) \neq \varnothing$. *Then*

(i) *problem* (1.19) *is well posed iff the operator* A *has the square root* $\sqrt{A}$ *generating a* C_0*-semigroup stable in degree* 2;

(ii) *problem* (1.20) *is well posed iff the operator* A *has the square root* $\sqrt{A}$ *generating a* C_0*-semigroup stable in degree* 1;

(iii) *problem* (1.21) *is well posed iff the operator* A *has the square root* $\sqrt{A}$ *generating a stable* C_0*-semigroup.*

Corollary 1.4.1. *Any operator* A *has not more than one square root* $\sqrt{A}$ *generating a* C_0*-semigroup stable in degree* 2.

Theorem 1.4.3. *Let* B *and* C *be self-adjoint commuting operators on a Hilbert space* H. *Then there exist closed complementable subspaces* H_1 *and* H_2 *such that if* $A = B + iC$, *then problems* (1.19), (1.20), *and* (1.21) *are well posed on* H_1 *and the problem*

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ u(0) = u^0, \quad u'(0) = u^1. \end{gathered} \tag{1.22}$$

is well posed on H_2.

Definition 1.4.2. The regularized *fractional derivative* of order $0 < \alpha < 1$ of a function $u(\cdot)$ is the function

$$(D^{(\alpha)}u)(t) = (D^{\alpha}_{0+}u)(t) - \frac{1}{\Gamma(1-\alpha)} \frac{u(0)}{t^{\alpha}},$$

where

$$(D^{\alpha}_{0+}u)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_0^t \frac{u(\xi)}{(t-\xi)^{\alpha}} \, d\xi \right).$$

Consider the Cauchy problem

$$(D^{(\alpha)}u)(t) = Au(t), \quad 0 < t \leq T, \; u(0) = u^0, \tag{1.23}$$

with a closed operator A. By a solution of problem (1.23) we mean a function $u(\cdot)$ such that

(i) $u(\cdot) \in C([0,T];E)$;

(ii) for $t \in \mathbb{R}_+$, the values $u(t) \in D(A)$;

(iii) the fractional integral $\dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \frac{u(\xi)}{(t-\xi)^\alpha}\, d\xi$ is continuously differentiable for $t \geq 0$, and

(iv) the function $u(\cdot)$ satisfies (1.23).

Theorem 1.4.4 ([44]). *Let there exist the resolvent* $(\lambda^\alpha I - A)^{-1}$ *for* $\lambda > \omega > 0$, *and let*

$$\overline{\lim_{\lambda\to\infty}} \lambda^{-1/\alpha} \ln \|(\lambda I - A)^{-1}\| = 0.$$

Then a solution of problem (1.23) *is unique.*

Theorem 1.4.5 ([44]). *Let the resolvent* $(\lambda^\alpha I - A)^{-1}$ *exist in the half-plane* $\operatorname{Re}\lambda > \omega > 0$, *and for the same* λ, *let*

$$\|(\lambda^\alpha I - A)^{-1}\| \leq C(1 + |\operatorname{Im}\lambda|)^{-\beta}, \quad 0 < \beta < 1.$$

Then problem (1.23) *has a unique solution. This solution is infinitely differentiable for* $t > 0$, *and for each* t, *its value continuously depends on the initial data* u^0.

Chapter 2

COSINE AND SINE OPERATOR FUNCTIONS

The existing parallelism between the theory of C_0-semigroups of operators and the theory of C_0-cosine operator functions has a distinctive character. On one hand, a number of definitions and properties practically repeat each other almost literally. On the other hand, for second order equations, by the Kisynski theorem, the main object corresponding to a C_0-cosine operator function is a C_0-group, which excludes the appearance of "parabolicity," despite the fact that the generator A of a C_0-cosine operator function also generates an analytic C_0-semigroup.

2.1. Measurability of Operator Semigroups and Cosine Operator Functions

The measurability property of a cosine operator function profitably differs from that of a semigroup. By the evenness, the measurability of a cosine operator function implies the strong continuity at zero. We have an analogous situation for perturbation families.

Definition 2.1.1. A function $T(\cdot) : \overline{\mathbb{R}}_+ \to B(E)$ is called an *operator semigroup* if it satisfies the conditions $T(t+h) = T(t)T(h)$ for any $t, h \in \overline{\mathbb{R}}_+$ and $T(0) = I$.

Definition 2.1.2. A function $C(\cdot) : \mathbb{R} \to B(E)$ is called an *operator cosine* (or a *cosine operator function)* if it satisfies the condition $C(t+h) + C(t-h) = 2C(t)C(h)$ for any $t, h \in \mathbb{R}$ and $C(0) = I$.

Definition 2.1.3. A function $S(\cdot) : \mathbb{R} \to B(E)$ is called an *operator sine* (or a *sine operator function*) if it satisfies the condition $S(t+h) + S(t-h) = 2S(t)C(h)$ for any $t, h \in \mathbb{R}$ and $S(0) = 0$.

Theorem 2.1.1 ([186]). *Let an operator semigroup $T(\cdot)$ be strongly measurable, i.e., the function $T(\cdot)x$ is strongly measurable on $\mathbb{R}_+$ for any $x \in E$. Then it is strongly continuous on $\mathbb{R}_+$.*

We stress that in Theorem 2.1.1, the strong continuity is asserted only on $\mathbb{R}_+$ but not on $\overline{\mathbb{R}}_+$!

Proposition 2.1.1 ([186]). *Let a function $t \to T(t)x$ be strongly measurable on $\mathbb{R}_+$. Then it is locally bounded.*

Proposition 2.1.2 ([186]). *Let an operator cosine $C(\cdot)$ be strongly measurable on $\mathbb{R}_+$. Then it is strongly continuous on $\mathbb{R}$.*

Proposition 2.1.3 ([186]). *Let a function $t \to C(t)x$ be strongly measurable on $\mathbb{R}_+$. Then it is locally bounded.*

Theorem 2.1.2 ([185]). *Let a cosine operator function $C(\cdot)$ be such that its restriction to a certain interval $J \subseteq \mathbb{R}$ be weakly Lebesgue measurable, and let the space E be separable and reflexive. Then $C(\cdot)$ is weakly continuous on $\mathbb{R}$.*

2.2. Multiplicative and Additive Families. Measurability and Continuity

Definition 2.2.1. Let $C(\cdot, A)$ be a C_0-cosine operator function. A family $\{F(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is called a *multiplicative perturbation family* for $C(\cdot, A)$ if $F(0) = 0$ and

$$F(t+s) - 2F(t) + F(t-s) = 2C(t, A)F(s) \quad \text{for} \quad t,\ s \in \mathbb{R}. \tag{2.1}$$

Definition 2.2.2. A family $\{G(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is called an *additive perturbation family* for a C_0-cosine operator function $C(\cdot, A)$ if $G(0) = 0$ and

$$G(t+s) - 2G(t) + G(t-s) = 2G(s)C(t, A) \quad \text{for} \quad t,\ s \in \mathbb{R}. \tag{2.2}$$

If these families are strongly continuous at zero, then they are called a *multiplicative perturbation C_0-family* and *additive perturbation C_0-family*, respectively.

Clearly, $F(\cdot)$ and $G(\cdot)$ are even functions. The terminology mentioned above is chosen by analogy with the corresponding definitions of perturbation families $U(\cdot)$ and $V(\cdot)$ for C_0-semigroups ([17, Sec. 2.2]).

Recall that $U(\cdot)$ satisfies the relations $U(0) = 0$ and $U(t+s) - U(t) = T(t)U(s)$, $t, s \in \overline{\mathbb{R}}_+$, and $V(\cdot)$ satisfies the relations $V(0) = 0$ and $V(t+s) - V(t) = V(s)T(t)$, $t, s \in \overline{\mathbb{R}}_+$.

The multiplicative and additive perturbation C_0-families play an important role in the perturbation theory of C_0-cosine operator functions.

For example, using the multiplicative and additive perturbation C_0-families, we can consider well-posed statements of the Cauchy problem in the form

$$u''(t) = A(1 - \lambda\hat{F}(\lambda))u(t) + \lambda^3\hat{F}(\lambda)u(t),\ t \in \overline{\mathbb{R}}_+, \quad u(0) = x,\ u'(0) = y.$$

As is known, a C_0-cosine operator function that is strongly (resp. uniformly) measurable on $\mathbb{R}_+$ is strongly (resp. uniformly) measurable on $\mathbb{R}$ (see Sec. 2.1). The following theorem shows that multiplicative and additive perturbation families have the same properties.

Theorem 2.2.1 ([239]). *If a multiplicative perturbation family $F(\cdot)$ is strongly (resp. uniformly) measurable on $\mathbb{R}_+$, then the function $F(\cdot)$ is strongly (resp. uniformly) continuous on $\mathbb{R}$.*

If an additive perturbation family $G(\cdot)$ is uniformly measurable on $\mathbb{R}_+$, then the function $G(\cdot)$ is uniformly continuous on $\mathbb{R}$.

Proof. First of all, the strong continuity of $F(\cdot)x$ on $\mathbb{R}_+$ implies the Lebesgue measurability of $\|F(\cdot)x\|$ on $\mathbb{R}_+$ (see [76]). Further, let us show that $\|F(\cdot)x\|$ is bounded on any compact subinterval $[a, b] \subset \mathbb{R}_+$ for any $x \in E$. Suppose the contrary. Then there exist $\tilde{x} \in E$, a number $\tau > 0$, and a sequence $\tau_n \in [a, b]$ such that $\tau_n \to \tau$ and

$$\|F(\tau_n)\tilde{x}\| \geq n \quad \text{as} \quad n \to \infty.$$

By the measurability of $\|F(\cdot)\tilde{x}\|$, there exist a constant c_1 and a Lebesgue measurable set $\Lambda \subset [0, \tau]$ of measure

$$\mu(\Lambda) > \frac{3}{4}\tau$$

such that

$$\sup_{t \in \Lambda} \|F(t)\tilde{x}\| \leq c_1. \tag{2.3}$$

Now, following [102], we set

$$\mathcal{A}_k := \frac{\tau_k}{2} - \frac{\Lambda \cap [0, \tau_k]}{2}, \quad \mathcal{B}_k := \Lambda \cap [0, \tau_k/2] \tag{2.4}$$

and

$$\mathcal{A} = \frac{\tau}{2} - \frac{\Lambda}{2}, \quad \mathcal{B} = \Lambda \cap [0, \tau/2].$$

First, $\mu(\mathcal{A}\cap\mathcal{B}) > 0$. To prove this, assume that $\mu(\mathcal{A}\cap\mathcal{B}) = 0$. Then $\mu(\mathcal{A})+\mu(\mathcal{B}) \leq \tau/2$. But $\mu(\mathcal{A}) = \mu/2(\Lambda)$ by the definition of the set $\mathcal{A}$. This means that $\mu(\Lambda)+2\mu(\mathcal{B}) \leq \tau$. Therefore, $\frac{3}{4}\tau < \mu(\Lambda) \leq \tau - 2\mu(\mathcal{B})$, i.e.,

$$\mu(\mathcal{B}) \leq \tau/8. \tag{2.5}$$

Write

$$\Lambda = (\Lambda \cap [0,\tau/2]) \ \cup \ (\Lambda \cap [\tau/2,\tau]) = \mathcal{B} \cup \mathcal{D},$$

where $\mu(\Lambda) = \mu(\mathcal{B}) + \mu(\mathcal{D})$ with $\mu(\mathcal{D}) \leq \tau/2$. But

$$\frac{3}{4}\tau < \mu(\Lambda) = \mu(\mathcal{B}) + \mu(\mathcal{D}) \leq \mu(\mathcal{B}) + \tau/2$$

implies $\mu(\mathcal{B}) > \tau/4$, which contradicts (2.5). We have proved that $\mu(\mathcal{A}\cap\mathcal{B}) \geq \delta > 0$.

Now define the sets

$$\mathcal{E} = \mathcal{A} \cap \mathcal{B}, \quad \mathcal{E}_n = \mathcal{A}_n \cap \mathcal{B}_n,$$

and

$$H_n = \{\tau_n - \eta, \quad \eta \in \mathcal{E}_n\}.$$

Clearly, $\mathcal{E}_n \to \mathcal{E}$ as $n \to \infty$, so that $\mu(H_n) > \delta/2$ for sufficiently large n. For the same n, if $\eta \in \mathcal{E}_n$, then η and $\tau_n - 2\eta$ belong to Λ by (2.4). Now, using (2.1) and (2.3), for $\eta \in \mathcal{E}_n$, we obtain

$$\begin{aligned} n \leq \|F(\tau_n)\tilde{x}\| &\leq 2\|F(\tau_n - \eta)\tilde{x}\| + \|F(\tau_n - 2\eta)\tilde{x}\| + 2\|C(\tau_n - \eta)\|\|F(\eta)\tilde{x}\| \\ &\leq 2\|F(\tau_n - \eta)\tilde{x}\| + c_1 + 2Me^{\omega b}c_1. \end{aligned}$$

Therefore,

$$\|F(t)\tilde{x}\| \geq \frac{n - c_1 - 2Mc_1e^{\omega\beta}}{2}$$

for $t \in H_n$; denoting $\lim_{n\to\infty} H_n = H_\infty$, we have $\|F(t)\tilde{x}\| = \infty$ for $t \in H_\infty$ with $\mu(H_\infty) \geq \delta/2 > 0$. This contradicts the boundedness of $\|F(t)\tilde{x}\|$ for each t.

We now want to prove that the strong measurability, together with the boundedness, implies the continuity of $F(\cdot)x$ for each $t \in \mathbb{R}_+$ and each $x \in E$. For this purpose, we choose four positive numbers α, β, ϵ, and γ such that $\beta < t - \epsilon$ and $0 < \alpha < \gamma < \beta < t$. We have from (2.1) that

$$F(t)x = 2F(t - \gamma/2)x - F(t-\gamma)x + 2C(t-\gamma/2, A)F(\gamma/2)x. \tag{2.6}$$

The left-hand side, being independent of γ, is integrable in γ, and we have

$$\begin{aligned}&(\beta-\alpha)(F(t\pm\epsilon)x-F(t)x)\\ &=\int_\alpha^\beta 2\Big(F(t\pm\epsilon-\gamma/2)-F(t-\gamma/2)x\Big)\,d\gamma-\int_\alpha^\beta\Big(F(t\pm\epsilon-\gamma)-F(t-\gamma)\Big)x\,d\gamma\\ &\quad+\int_\alpha^\beta 2\Big(C(t\pm\epsilon-\gamma/2,A)-C(t-\gamma/2,A)\Big)F(\gamma/2)x\,d\gamma.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|(F(t\pm\epsilon)-F(t))x\|\\ &\leq\frac{1}{\beta-\alpha}\Bigg[\int_{t-\beta/2}^{t-\alpha/2}\|(F(\zeta\pm\epsilon)-F(\zeta))x\|d\zeta+\int_{t-\beta}^{t-\alpha}\|(F(\zeta\pm\epsilon)-F(\zeta))x\|d\zeta\\ &\quad+2\int_\alpha^\beta\|(C(t\pm\epsilon-\gamma/2,A)-C(t-\gamma/2,A))F(\gamma/2)x\|\,d\gamma\Bigg].\end{aligned}\tag{2.7}$$

By Theorem 3.8.3 from [76],

$$\int_{t-\beta/2}^{t-\alpha/2}\to 0\quad\text{and}\quad\int_{t-\beta}^{t-\alpha}\to 0\quad\text{as}\quad\epsilon\to 0.$$

The last summand in (2.7) tends to zero by the Lebesgue theorem on the dominated convergence (see [76, Theorem 3.7.9]).

We obtain that $F(\cdot)x$ is continuous for $t\in\mathbb{R}_+$. Replacing t by $t+s$ in (2.1), we obtain that for all $t,\ s\in\mathbb{R}_+$, the function

$$F(t)x=2C(t+s,A)F(s)x-F(t+2s)x+2F(t+s)x$$

tends to $2C(s,A)F(s)x-F(2s)x+2F(s)x=F(0)x=0$ as $t\to 0+$. Therefore, $F(\cdot)$ is strongly continuous on $\overline{\mathbb{R}}_+$ and hence on $\mathbb{R}$, since $F(\cdot)$ is an even function. The proof for the case of uniform measurability is analogous.

To prove the assertion for $G(\cdot)$, we can use the following writing of Eq. (2.2):

$$G(\tau_n)=2G(\tau_n-\eta)-G(\tau_n-2\eta)+2G(\eta)C(\tau_n-\eta,A)$$

in operating the estimate of form (2.6). The proof is analogous.

Theorem 2.2.2 ([239]). *A multiplicative perturbation C_0-family and an additive perturbation C_0-family are strongly continuous on $\overline{\mathbb{R}}_+$ for a C_0-cosine operator function $C(\cdot,A)$. Moreover, the uniform continuity at 0 implies the uniform continuity on $\overline{\mathbb{R}}_+$.*

Proof. Following [102], we assume the contrary: the multiplicative perturbation family $F(\cdot)$ is not strongly continuous at a certain point $t_0 \in \mathbb{R}_+$, i.e., there exists x_0 such that the nonincreasing sequence

$$K_n := \sup\left\{ \|(F(t) - F(s))x_0\| : |t - t_0|, |s - t_0| \leq \frac{t_0}{8n} \right\}$$

converges to a certain $K > 0$ as $n \to \infty$.

We can choose sequences τ_n and σ_n such that

$$|\tau_n - t_0| \leq \frac{t_0}{8n}, \quad |\sigma_n - t_0| \leq \frac{t_0}{8n},$$

and

$$\|(F(\tau_n) - F(\sigma_n))x_0\| \geq K_n - \frac{1}{n}, \quad n \in \mathbb{N}.$$

Clearly, $|\sigma_n - \tau_n| \leq \frac{t_0}{4n}$ and $|2\tau_{4n} - \sigma_{4n} - t_0| \leq \frac{t_0}{8n}$, $n \in \mathbb{N}$. Therefore,

$$\|(F(\sigma_{4n}) - F(2\tau_{4n} - \sigma_{4n}))x_0\| \leq K_n, \quad n \in \mathbb{N}.$$

Using identity (2.1) in the form

$$2(F(t+h) - F(t)) = (F(t+h) - F(t-h)) + 2C(t, A)F(h)$$

and setting $t_0 + h = \sigma_{4n}$ and $t_0 = \tau_{4n}$, we obtain

$$2\|(F(\sigma_{4n}) - F(\tau_{4n}))x_0\| \leq K_n + 2Me^{\omega t_0}\|F(\sigma_{4n} - \tau_{4n})x_0\|.$$

Therefore,

$$2\left(K_{4n} - \frac{1}{4n}\right) \leq K_n + 2Me^{\omega t_0}\|F(\sigma_{4n} - \tau_{4n})x_0\|,$$

and, thus,

$$K_{4n} + (K_{4n} - K_n) \leq \frac{1}{2n} + 2Me^{\omega t_0}\|F(h)x_0\|.$$

By the convergence $F(h)x_0 \to 0$ as $h \to 0$ (recall that $h = \sigma_{4n} - \tau_{4n}$) and $K_{4n} - K_n \to 0$ as $n \to \infty$ we have $K_n \to 0$ as $n \to \infty$, $n \in \mathbb{N}$, which is a contradiction to our assumption that $K_n \to K,\ K > 0$.

To prove the same assertion for $G(\cdot)$, we can use the identity

$$2(G(t+h) - G(t)) = (G(t+h) - G(t-h)) + 2G(t)(C(h, A) - I) + 2G(h),$$

which is obtained from (2.2) and Proposition 2.4.1 (i).

In the same way as in Proposition 2.3.2 in [17], we can prove the following assertion.

Proposition 2.2.1. *Let a multiplicative perturbation family $F(\cdot)$ and a C_0-cosine operator function $C(\cdot, A)$ commute, i.e., $F(t)C(t, A) = C(t, A)F(t)$ for all $t \in \overline{\mathbb{R}}_+$. Then the multiplicative perturbation C_0-family $F(\cdot)$ is an additive perturbation family and it is commutative, i.e., $F(t)F(s) = F(s)F(t)$ for all $s, t \in \mathbb{R}$.*

2.3. Main Properties of C_0-Cosine and C_0-Sine Operator Functions

Definition 2.3.1. A *C_0-cosine operator function* is defined as a one-parameter family of operators $\{C(t),\ t \in \mathbb{R}\}$, $C(t) \in B(E)$, $t \in \mathbb{R}$, having the following properties:

(i) $C(t + s) + C(t - s) = 2C(t)C(s)$ for any $t, s \in \mathbb{R}$ (d'Alembert equation);

(ii) $C(0) = I$ is the identity operator on E;

(iii) $s\text{-}\lim\limits_{h \to 0} C(h)x = x$ for any $x \in E$.

With a C_0-cosine operator function $C(\cdot)$, we associate the *C_0-sine operator function*

$$S(t)x := \int_0^t C(s)x\, ds, \qquad x \in E,\ t \in \mathbb{R}, \tag{2.8}$$

and the lineals

$$E^k := \{x \in E :\ C(\cdot)x \in C^k(\mathbb{R}; E)\}, \qquad k = 1, 2. \tag{2.9}$$

Definition 2.3.2. A linear operator A with the domain $D(A)$ consisting of all x for which there exists the limit

$$Ax := s\text{-}\lim_{h \to 0+} 2\frac{C(h) - I}{h^2} x \tag{2.10}$$

is called an *infinitesimal generator* of a C_0-cosine function $C(\cdot)$.

The property that A is a generator of a C_0-cosine operator function $C(\cdot)$ is written as $C(\cdot, A)$ (and $S(\cdot, A)$ for a C_0-sine operator function $S(\cdot)$).

Let us present a simplest example of a C_0-cosine operator function.

Example 2.3.1. Let A be the operator of multiplication by a complex number on the space $\mathbb{R}$. Then A is the generator of the C_0-cosine operator function $(C(t, A)f)(s) = \cos(it\sqrt{A})f(s)$, $t \in \mathbb{R}$.

Proposition 2.3.1 ([88]). *Define the operator*

$$A_1x := s\text{-}\lim_{h \to 0} \frac{C(2h, A) - 2C(h, A) + I}{h^2} x$$

with a natural domain (i.e., on those $x \in E$ at which this limit does exist). Then for $x \in D(A_1) \cap D(A)$, we have $Ax = A_1x$.

For a C_0-cosine operator function $C(\cdot, A)$, we can also define the first generator

$$\overset{\circ}{\mathcal{C}}x := s\text{-}\lim_{h\to 0+} \frac{C(h, A)x - x}{h}$$

with a natural domain.

Proposition 2.3.2 ([264]). *For a C_0-cosine operator function $C(\cdot, A)$, we have $D(A) \subseteq D(\overset{\circ}{\mathcal{C}})$ and $\overset{\circ}{\mathcal{C}}x = 0$ for any $x \in D(A)$.*

Proposition 2.3.3 ([264]). *The operators $C(t, A), C(s, A), S(t, A)$, and $S(s, A)$ commute for any $t, s \in \mathbb{R}$.*

Proposition 2.3.4. *The C_0-sine operator function $S(\cdot, A)$ is continuous in the uniform operator topology.*

Proposition 2.3.5 ([264, 272]). *For all $t, s \in \mathbb{R}$, we have the relations*

$$\begin{aligned}
&\text{(i)} \quad C(t, A) = C(-t, A), \quad S(-t, A) = -S(t, A), \quad S(0, A) = 0;\\
&\text{(ii)} \quad S(t+s, A) + S(t-s, A) = 2S(t, A)C(s, A);\\
&\text{(iii)} \quad S(t+s, A) = S(t, A)C(s, A) + S(s, A)C(t, A);\\
&\text{(iv)} \quad C(t+s, A) - C(t-s, A) = 2AS(t, A)S(s, A);\\
&\text{(v)} \quad C(2t, A) = 2C(t, A)^2 - I, \qquad C(t, A)^2 - AS(t, A)^2 = I;\\
&\text{(vi)} \quad C((n+1)t, A) = b_0 I + b_1 C(t, A) + ... + b_{n+1} C^{n+1}(t, A),
\end{aligned}$$

where $b_0 + b_1 z + ... + b_{n+1} z^{n+1}$ is the Chebyshev polynomial of the first kind of degree $n+1$.

Proposition 2.3.6 ([264]). *For any C_0-cosine operator function $C(\cdot, A)$, there exist constants $M \geq 1$ and $\omega \geq 0$ such that for all $t \in \mathbb{R}$, we have the estimate*

$$\|C(t, A)\| \leq M \cosh(\omega t), \quad t \in \mathbb{R}, \tag{2.11}$$

where $\cosh(\omega t) := \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$ is the hyperbolic cosine.

Definition 2.3.3. Infimum of the numbers ω from (2.11) is called the *type* of a C_0-cosine operator function and is denoted by $\omega_c(A)$.

Proposition 2.3.7 ([207]). *The minimum ω satisfying (2.11) for an appropriate constant M_ω may not exist, i.e., the greatest lower bound of $\omega_c(A)$ is not attained in general.*

Proposition 2.3.8 ([133, 142, 264]). *Let an operator A generate a C_0-cosine operator function $C(t, A)$, and let $\|C(t, A)\| \le M \cosh(\omega t)$, $t \in \mathbb{R}$. Then $A \in \mathcal{G}(M, \omega^2)$, the C_0-semigroup $\exp(\cdot A)$ is analytically continued to the right half-plane, and*

$$\exp(tA) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} C(s, A) ds, \quad t \in \overline{\mathbb{R}}_+. \tag{2.12}$$

Proposition 2.3.9. *The representation of the analytic semigroup in Proposition 2.3.8 can be written in the form*

$$\exp(tA)x = \frac{1}{2^k \sqrt{\pi} t^{(k+1)/2}} \int_0^\infty P_k \left(\frac{s}{2\sqrt{t}} \right) e^{-\frac{s^2}{4t}} C_k(s) x \, ds, \quad t \in \mathbb{R}_+.$$

Here, P_k is a polynomial of degree k and $C_k(t) = \int\limits_0^t \frac{(t-s)^{k-1}}{(k-1)!} C(s, A) \, ds$, where $t \in \overline{\mathbb{R}_+}$, $k \in \mathbb{N}$, and $C_0(t) = C(t, A)$.

Remark 2.3.1 ([223]). There are examples of analytic C_0-semigroups whose generators do not generate C_0-cosine operator functions.

Proposition 2.3.10 ([179]). *Obviously, $D(A) \subseteq E^1$ for $A \in \mathcal{C}(M, \omega)$, and, therefore, the set E^1 is dense in E.*

Proposition 2.3.11 ([274]). *For any $x \in E$ and $t, s \in \mathbb{R}$, we have*

$$\text{(i)} \quad y := \int_s^t S(\tau, A) x d\tau \in D(A) \quad \textit{and} \quad Ay = C(t, A)x - C(s, A)x; \tag{2.13}$$

$$\text{(ii)} \quad z := \int_0^t \int_0^s C(\tau, A) C(\zeta, A) x d\tau d\zeta \in D(A) \quad \textit{and} \tag{2.14}$$

$$Az = \frac{1}{2} \Big(C(t+s, A) - C(t-s, A) \Big) x; \tag{2.15}$$

$$\text{(iii)} \quad S(t, A)x \in E^1. \tag{2.16}$$

Proposition 2.3.12 ([274]). *If elements x vary over the whole E, and the numbers t and s vary over $\mathbb{R}$, then the set of elements of the form $y = \int\limits_s^t S(\tau, A) x d\tau$ is dense in E.*

Proposition 2.3.13. *For any $x \in E$, the following relations hold:*

$$s\text{-}\lim_{t \to 0} t^{-1} S(t, A)x = x \quad \textit{and} \quad s\text{-}\lim_{t \to 0} 2t^{-2} \int_0^t S(\tau, A) x d\tau = x. \tag{2.17}$$

Proposition 2.3.14 ([274]). *If $x \in E^1$, then for any $t \in \mathbb{R}$,*

$$\text{(i)} \quad C(t,A)x \in E^1, \quad S(t,A)x \in D(A) \quad \text{and} \quad C'(t,A)x = AS(t,A)x; \tag{2.18}$$

$$\text{(ii)} \quad s\text{-}\lim_{\tau \to 0} AS(\tau,A)x = 0 \quad \text{and} \quad S''(t,A)x = AS(t,A)x. \tag{2.19}$$

Proposition 2.3.15 ([274]). *Let $x \in D(A)$. Then for all $t \in \mathbb{R}$,*

$$\text{(i)} \quad C(t,A)x \in D(A) \quad \text{and} \quad C''(t,A)x = AC(t,A)x = C(t,A)Ax; \tag{2.20}$$

$$\text{(ii)} \quad S(t,A)x \in D(A) \quad \text{and} \quad S''(t,A)x = AS(t,A)x = S(t,A)Ax. \tag{2.21}$$

Proposition 2.3.16 ([272]). *For all $t, s \in \mathbb{R}$, the following relations hold:*

$$\text{(i)} \quad C(2t,A) = C(t,A)^2 + C'(t,A)S(t,A); \tag{2.22}$$

$$\text{(ii)} \quad C'(t,A)S(s,A) = C'(s,A)S(t,A); \tag{2.23}$$

$$\text{(iii)} \quad C(t+s,A) - C(t-s,A) = 2C'(t,A)S(s,A); \tag{2.24}$$

$$\text{(iv)} \quad (C(t,A) - I)\int_0^h S(s,A)ds = (C(h,A) - I)\int_0^t S(s,A)ds; \tag{2.25}$$

$$\text{(v)} \quad (A - \lambda^2 I)\int_0^t \sinh\Big(\lambda(t-s)\Big)C(s,A)ds = \lambda\Big(C(t,A) - \cosh(\lambda t)I\Big); \tag{2.26}$$

here, $\sinh(\cdot)$ *and* $\cosh(\cdot)$ *are the hyperbolic sine and the hyperbolic cosine, respectively.*

Proposition 2.3.17 ([135]). *The domain of the generator of a C_0-cosine function $C(\cdot, A)$ coincides with E^2, and for each $x \in D(A)$,*

$$Ax = s\text{-}\lim_{\tau \to 0} C''(\tau,A)x. \tag{2.27}$$

Sometimes the generator of a C_0-cosine operator function is defined by (2.27).

The set of generators of a C_0-cosine operator function with bound (2.11) will be denoted by $\mathcal{C}(M,\omega)$.

Proposition 2.3.18 ([264]). *Let $A, G \in \mathcal{C}(M,\omega)$. Then if $D(A) \subseteq D(G)$ and $Ax = Gx$ for all $x \in D(A)$, we have $C(t,A) = C(t,G)$ for all $t \in \mathbb{R}$.*

Theorem 2.3.1 ([113, 131, 264, 269]). *For an operator $A \in \mathcal{C}(E)$ to be a generator of a C_0-cosine operator function, it is necessary and sufficient that for a certain constants $M, \omega \geq 0$, the resolvent $(\lambda^2 I - A)^{-1}$ exists for $\operatorname{Re}\lambda > \omega$ and the following inequalities hold:*

$$\left\| \frac{d^n}{d\lambda^n}\Big(\lambda(\lambda^2 I - A)^{-1}\Big) \right\| \leq \frac{Mn!}{(\operatorname{Re}\lambda - \omega)^{n+1}}, \qquad n \in \mathbb{N}_0. \tag{2.28}$$

Remark 2.3.2 ([264])**.** Sometimes, estimate (2.28) is written in the form

$$\left\| \frac{d^n}{d\lambda^n}\left(\lambda(\lambda^2 I - A)^{-1}\right)\right\| \leq \frac{Mn!}{2}\left(\frac{1}{(\operatorname{Re}\lambda - \omega)^{n+1}} + \frac{1}{(\operatorname{Re}\lambda + \omega)^{n+1}}\right) \tag{2.29}$$

for all $\operatorname{Re}\lambda > \omega, n \in \mathbb{N}_0$.

In practice, conditions (2.28)–(2.29) turn out to be difficult to verify, and, therefore, other conditions for generating a C_0-cosine operator function are of interest.

Theorem 2.3.2 ([42])**.** *An operator $A \in \mathcal{C}(E)$ is a generator of a C_0-cosine operator function iff there exist constants $M, \delta > 0$, and ω such that*

$$\|(\lambda^2 I - A)^{-1}\| \leq \frac{M}{|\lambda|(\operatorname{Re}\lambda - \omega)} \quad \text{for all} \quad \operatorname{Re}\lambda > \omega, \tag{2.30}$$

and the following estimate holds uniformly in $\tau \in (0, \delta)$:

$$\left\| \int\limits_{\omega - i\infty}^{\omega + i\infty} e^{\lambda^2 \tau} \cosh(\lambda t)\lambda(\lambda^2 I - A)^{-1} x \, d\lambda \right\| \leq \xi(t)\|x\|, \quad t \in \overline{\mathbb{R}_+}, \tag{2.31}$$

where $\xi(\cdot) \in C(\mathbb{R})$.

Remark 2.3.3. In connection with estimate (2.30), we note (see [57]) that for any fixed $\varepsilon > 0$, the condition $\|(\lambda^2 I - A)^{-1}\| \leq \dfrac{M}{|\lambda|^{1+\varepsilon}}$, $\operatorname{Re}\lambda > \omega$, implies the boundedness of the spectrum $\sigma(A)$.

Proposition 2.3.19 ([210])**.** *In the case where A is a normal operator on a Hilbert space, it generates a C_0-cosine operator function if and only if the conditions for location of the spectrum hold, i.e., $\{z^2 : \operatorname{Re} z > \omega\} \subseteq \rho(A)$ for a certain ω.*

Proposition 2.3.20 ([272])**.** *For $\operatorname{Re}\lambda > \omega_c(A)$, we have $\lambda^2 \in \rho(A)$ and*

$$\lambda(\lambda^2 I - A)^{-1} x = \int_0^\infty e^{-\lambda t} C(t, A) x dt, \quad x \in E; \tag{2.32}$$

$$(\lambda^2 I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t, A) x dt, \quad x \in E. \tag{2.33}$$

Proposition 2.3.21 ([135])**.** *If $x \in D(A^3)$, $y \in D(A)$, and $\omega > \omega_c(A)$, then*

$$C(t, A)x = x + \frac{t^2}{2!}Ax + \frac{t^4}{4!}A^2 x + \frac{1}{2\pi i}\int\limits_{\omega - i\infty}^{\omega + i\infty} e^{\lambda t}\lambda^{-3}(\lambda^2 I - A)^{-1} A^3 x \, d\lambda; \tag{2.34}$$

$$C(t, A)y = \frac{1}{2\pi i}\int\limits_{\omega - i\infty}^{\omega + i\infty} e^{\lambda t}\lambda(\lambda^2 I - A)^{-1} y \, d\lambda, \ t \in \mathbb{R}_+. \tag{2.35}$$

Writing the inverse Laplace transform in another form, we can obtain other analogous representations of the operator functions $C(\cdot, A)$ and $S(\cdot, A)$.

Proposition 2.3.22 ([222]). *Let $x \in D(A^k)$ for a certain $k \in \mathbb{N}$. Then for $t \in \mathbb{R}$, the following Taylor formula holds*:

$$C(t,A)x = x + \frac{t^2}{2!}Ax + \dots + \frac{t^{2k-2}}{(2k-2)!}A^{k-1}x + \int_0^t \frac{(t-s)^{2k-1}}{(2k-1)!}C(s,A)A^k x ds.$$

Proposition 2.3.23 ([166]). *Let $A \in \mathcal{C}(M,\omega)$, and let $r \in \mathbb{N}$. Then*

$$\Big(C(t,A) - I\Big)^r = 2^{-r}\left[2\sum_{j=1}^{r}(-1)^{r-j}C_{r-j}^{2r}C(jt,A) + (-1)^r C_r^{2r} I\right],$$

$$\Big(C(t,A) - I\Big)^r x = A^r \int_0^t \int_0^t \dots \int_0^t \prod_{j=1}^{r}(t-s_j)C(s_j,A)x \, ds_1 ds_2 \dots ds_r$$

for any $x \in E$.

Proposition 2.3.24 ([225]). *For any $A \in \mathcal{C}(M,\omega)$, $x \in \tilde{E}_0$, and $t \in \mathbb{R}$ the representation*

$$C(t,A)x = \sum_{k=0}^{\infty} t^{2k}A^k x/(2k)!$$

holds, and for each $\tilde{x} \in \tilde{E}_0$, the function $t \to C(t,A)\tilde{x}$ can be continued in t up to a function analytic on the whole complex plane.

Proposition 2.3.25 ([133]). *The following Widder–Post formula holds*:

$$C(t,A)x = \lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1}\frac{d^k}{d\lambda^k}(\lambda(\lambda^2 I - A)^{-1}x)\Big|_{\lambda=\frac{k}{t}}, \quad t \neq 0, \; x \in E, \tag{2.36}$$

where the convergence is uniform in t from any compact set in $\mathbb{R} \setminus \{0\}$.

Proposition 2.3.26 ([289]). *The expression $\mathfrak{N}(\lambda,k) := \dfrac{d^k}{d\lambda^k}\Big(\lambda(\lambda^2 I - A)^{-1}\Big)$ from* (2.36) *can be represented in the form*

(i) $\mathfrak{N}(\lambda,k) = k!\Big(\lambda^{k+1} + C_{k+1}^2\lambda^{k-1}A + \dots + C_{k+1}^k\lambda A^{k/2}\Big)(\lambda^2 I - A)^{-(k+1)}, \quad k$ *is even;*

(ii) $\mathfrak{N}(\lambda,k) = -k!\Big(\lambda^{k+1} + C_{k+1}^2\lambda^{k-1}A + \dots + C_{k+1}^k\lambda A^{(k+1)/2}\Big)(\lambda^2 I - A)^{-(k+1)}, \quad k$ *is odd;*

(iii) $\mathfrak{N}(\lambda,k) = \displaystyle\sum_{j=k/2}^{k}(-1)^j\frac{(k+1)!j!\lambda(2\lambda)^{2j-k}}{(k-j)!(2j-k+1)!}(\lambda^2 I - A)^{-(j+1)}, \quad k$ *is even;*

(iv) $\mathfrak{N}(\lambda,k) = \displaystyle\sum_{j=\frac{k-1}{2}+1}^{k}(-1)^j\frac{(k+1)!j!\lambda(2\lambda)^{2j-k}}{(k-j)!(2j-k+1)!}(\lambda^2 I - A)^{-(j+1)}, \quad k$ *is odd.*

Proposition 2.3.27 ([260]). *For a C_0-cosine operator function $C(\cdot, A)$, C_0-sine operator function $S(\cdot, A)$, and any $x \in E$ and $t \in \mathbb{R}$, we have*

$$\text{(i)} \quad C(t, A)x = \lim_{k\to\infty} \sum_{l=0}^{k}\sum_{j=0}^{l} C_{2k}^{2l} C_l^j (-1)^{l-j} \left(I - \left(\frac{t}{2k}\right)^2 A \right)^{-(2k-l+j)} x;$$

$$\text{(ii)} \quad C(t, A)x = \lim_{k\to\infty} \sum_{l=0}^{k}\sum_{j=0}^{l} C_{2k+1}^{2l} C_l^j (-1)^{l-j} \left(I - \left(\frac{t}{2k+1}\right)^2 A \right)^{-(2k+1-l+j)} x;$$

$$\text{(iii)} \quad C(t, A)x = \lim_{n\to\infty} e^{-nt} \sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{k} \frac{(nt)^{2m}}{(2m)!} C_{2m}^{2k} C_k^j (-1)^{k-j}$$
$$\times \left(I + \frac{nt}{2m-2k+1} (I - n^{-2}A)^{-1} \right) (I - n^2 A)^{-(2m-k+j)} x;$$

$$\text{(iv)} \quad S(t, A)x = \lim_{n\to\infty} \frac{t}{n} \sum_{m=0}^{n-1}\sum_{k=0}^{m}\sum_{j=0}^{k} C_{2m+1}^{2k} C_k^j (-1)^{k-j} \left(I - \left(\frac{t}{2n}\right)^2 A \right)^{-(n+m+1-k+j)} x;$$

moreover, in all the cases, the convergence in $t \in J \subset \mathbb{R}$ is uniform. Here, J is an arbitrary closed interval.

Proposition 2.3.28 ([260]). *Under the conditions of Proposition* 2.3.27, *we have the following uniformly in $t \in [0, 1]$:*

$$C(t, A)x = \lim_{n\to\infty} \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{k} C_{2n}^{2m} C_{2m}^{2k} C_k^j (-1)^{k-j} t^{2m} (1-t)^{2n-2m-1}$$
$$\times \left((1-t) + \frac{2n-2m}{2m-2k+1} t (I - (2n)^{-2} A)^{-1} \right)^{-(2m-k+j)} x.$$

In [260], many other relations in an analogous form were also presented.

Introduce the following notation:

$$\mathrm{St}(A) := \left\{ x \in E : \sum_{k=1}^{\infty} \|A^k x\|^{-\frac{1}{2k}} < \infty \right\} \quad \text{are the Stieltjes vectors,}$$

$$\mathfrak{U}_p(A) := \left\{ x \in E : \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} \|A^k x\| < \infty \quad \text{for a certain} \quad t \in \mathbb{R}_+ \right\}$$

are semianalytic vectors, and

$$\mathfrak{U}_{pp}(A) := \left\{ x \in E : \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \|A^k x\| < \infty \quad \text{for all} \quad t \in \mathbb{R}_+ \right\} \quad \text{are entire vectors.}$$

Proposition 2.3.29 ([107]). *Let $A \in \mathcal{C}(M, \omega)$, and let $\tilde{E}_0$ be constructed according to the C_0-semigroup* $\exp(\cdot A)$. *Then $\tilde{E}_0 \subseteq \mathfrak{U}_p(A)$.*

Proposition 2.3.30 ([107]). *Let $A \in \mathcal{C}(M, \omega)$. We have the embeddings $\mathfrak{U}(A) \subseteq \mathfrak{U}_p(A) \subseteq \mathrm{St}(A)$.*

Proposition 2.3.31 ([107]). *Let $\overline{\mathfrak{U}_{pp}(A)} = E$. Then the set of vectors x from $D(A^\infty)$ having the property $\|A^k x\|^{1/k} = o(k)$ is dense in E.*

Proposition 2.3.32 ([107]). *Let $A \in \mathcal{C}(M, \omega)$. Then $\overline{\mathfrak{U}(A) \cap \mathfrak{U}_{pp}(A)} = E$.*

Proposition 2.3.33 ([107]). *Let $\overline{\mathfrak{U}_{pp}(A)} = E$ and let there exist an operator $G \in \mathcal{C}(E)$ such that*

(i) $G^{-1} \in B(E)$;

(ii) $G^2 = A$, *and, moreover, the operators $\pm G$ are dissipative.*

Then $A \in \mathcal{C}(M, \omega)$ and $C(t, A) = (exp(tG) + exp(-tG))/2$, where $G \in \mathcal{GR}(1, 0)$.

Definition 2.3.4. A set of elements $S \subseteq E$ is said to be *total* in E if the set of all its finite linear combinations is dense in E.

Proposition 2.3.34 ([107]). *Let $A_1 \in L(E)$ be closed, $\mathrm{St}(A_1)$ be total in E, $A_2 \in \mathcal{C}(M, \omega)$, and let $A_1 \subseteq A_2$. Then $A_1 = A_2$.*

Proposition 2.3.35 ([107]). *Let A be a closed, symmetric, and semibounded operator on a Hilbert space H. Then the operator A is self-adjoint iff the set $\mathrm{St}(A)$ is total in H.*

In [145], examples of nonlinear cosine operator functions are presented. However, in contrast to the theory of nonlinear operator semigroups, there is no general theory of nonlinear cosine operator functions for now.

2.4. Laplace Transform and Infinitesimal Operators

In this section, we present certain basic properties of the Laplace transform for C_0-families of multiplicative perturbations $F(\cdot)$ and additive perturbations $G(\cdot)$. Let $\hat{F}(\cdot)$ and $\hat{G}(\cdot)$ denote their Laplace transforms, respectively.

Proposition 2.4.1 ([239]). *Let $F(\cdot)$ be a C_0-family of multiplicative perturbations and let $G(\cdot)$ be a family of additive perturbations for a C_0-cosine operator function $C(\cdot, A)$. The following properties hold:*

(i) $(C(t, A) - I)F(s) = (C(s, A) - I)F(t)$ *and*

$G(s)(C(t, A) - I) = G(t)(C(s, A) - I)$ *for* $t, s \in \overline{\mathbb{R}}_+$;

(ii) *the functions $F(\cdot)$ and $G(\cdot)$ are exponentially bounded;*

(iii) $\dfrac{d^2}{dt^2}(\lambda(\lambda^2 I - A)^{-1} F(t)x) = C(t, A)\lambda^2 \hat{F}(\lambda)x$ *and*

$\dfrac{d^2}{dt^2}\Big(G(t)\lambda(\lambda^2 I - A)^{-1}x\Big) = \lambda^2 \hat{G}(\lambda)C(t, A)x$ *for $x \in E$, $\lambda > \omega$, and $t \in \mathbb{R}_+$;*

(iv)

$$F(t)x = (\lambda^2 I - A)\int_0^t S(s,A)\lambda\hat{F}(\lambda)x\,ds = \int_0^t S(s,A)\lambda^3\hat{F}(\lambda)x\,ds - (C(t,A)-I)\lambda\hat{F}(\lambda)x$$

for $x \in E,\ t \in \overline{\mathbb{R}}_+$;

(v)

$$G(t)x = \lambda\hat{G}(\lambda)(\lambda^2 I - A)\int_0^t S(s,A)x\,ds = \lambda^3\hat{G}(\lambda)\int_0^t S(s,A)x\,ds - \lambda\hat{G}(\lambda)(C(t,A)-I)x$$

for $\in E,\ t \in \overline{\mathbb{R}}_+$.

Proof. Property (i) is easily implied by (2.1) and (2.2). To prove that the C_0-family of multiplicative perturbations $F(\cdot)$ is exponentially bounded, we choose $L \geq 1$ and $\tau \in \mathbb{R}_+$ such that $\|C(s,A)\| \leq L$ and $\|F(s)\| \leq L$ for $0 \leq s \leq \tau$. Using the relation

$$F(k\tau + s) = 2F(k\tau) - F(k\tau - s) + 2C(k\tau, A)F(s)$$

for $0 \leq s \leq \tau$, we have

$$\begin{aligned}\|F(\tau+s)\| &\leq \|2F(\tau)\| + \|F(\tau - s)\| + \ 2\ Me^{\tau\omega}\|F(s)\| \\ &\leq 2L + L + 2Me^{\tau\omega}L \leq Me^{\tau\omega}5L \leq Me^{2\tau\omega_1} \\ &\leq Me^{\tau\omega_1}e^{(\tau+s)\omega_1},\end{aligned}$$

where $5L \leq e^{\tau\omega_1}$ and $\omega \leq \omega_1$,

$$\begin{aligned}&\|F(2\tau+s)\| \leq 2\|F(2\tau)\| + \|F(2\tau - s)\| + 2Me^{2\tau\omega_1}\|F(s)\| \\ &\leq 2Me^{2\tau\omega_1} + Me^{2\tau\omega_1} + 2LMe^{2\tau\omega_1} \leq Me^{3\tau\omega_1} \leq \tau\omega_1 e^{(2\tau+s)\omega_1}.\end{aligned}$$

By induction,

$$\begin{aligned}\|F(k\tau+s)\| &\leq 2\|F(k\tau)\| + \|F(k\tau - s)\| + 2\|C(k\tau, A)\|\ \|F(s)\| \\ &\leq 2Me^{k\tau\omega_1} + Me^{k\tau\omega_1} + 2LMe^{k\tau\omega_1} \leq 5LMe^{k\tau\omega_1} \\ &\leq Me^{(k+1)\tau\omega_1} \leq Me^{\tau\omega_1}e^{(k\tau+s)\omega_1}\end{aligned}$$

for all $s \in [0,\tau]$. Therefore, $\|F(t)\| \leq M_1 e^{\omega_1 t}$ for $M_1 = Me^{\tau\omega_1}$ and all $t \in \overline{\mathbb{R}}_+$.

To prove (iii), we set $\Theta(t,\lambda) = \lambda(\lambda^2 - A)^{-1}F(t)$ and $\Upsilon(t,\lambda) = G(t)\lambda(\lambda^2 - A)^{-1}, \lambda > \omega, t \geq 0$. It follows from (2.1) and (2.2) that

$$\Theta_t''(t,\lambda) = C(t,A)\lim_{s\to 0} 2s^{-2}\lambda(\lambda^2 - A)^{-1}F(s) = C(t,A)\Theta_t''(0,\lambda)$$

if $\Theta_t''(0,\lambda)$ exists, and

$$\Upsilon_t''(t,\lambda) = \lim_{s\to 0} 2s^{-2}\lambda(\lambda^2 - A)^{-1}G(s)C(t,A) = \Upsilon_t''(0,\lambda)C(t,A)$$

if $\Upsilon_t''(0,\lambda)$ exists. Therefore, it suffices to prove that $\Theta_t''(0,\lambda) = \lambda^2\hat{F}(\lambda)$ and $\Upsilon_t''(0,\lambda) = \lambda^2\hat{G}(\lambda)$.

Taking the Laplace transform in t in (2.1), we have

$$(e^{\lambda s} - 2 + e^{-\lambda s})\hat{F}(\lambda) - e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau + e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau = 2\lambda(\lambda^2 - A)^{-1}F(s) = 2\Theta(s,\lambda).$$

Now, taking the derivative, we obtain

$$2\Theta_s'(s,\lambda) = \lambda(e^{\lambda s} - e^{-\lambda s})\hat{F}(\lambda) - \lambda e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau - \lambda e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau,$$

and differentiating once more, we have

$$2\Theta_{ss}''(s,\lambda) = \lambda^2(e^{\lambda s} + e^{-\lambda s})\hat{F}(\lambda) - \lambda^2 e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau + \lambda^2 e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau - 2\lambda F(s).$$

Therefore, $\Theta_s'(0,\lambda) = 0$ and $\Theta_{ss}''(0,\lambda) = \lambda^2\hat{F}(\lambda)$. Analogously, we can show that $\Upsilon_s'(0,\lambda) = 0$ and $\Upsilon_{ss}''(0,\lambda) = \lambda^2\hat{G}(\lambda)$.

Integrating $\Theta_{tt}''(t,\lambda) = C(t,A)\lambda^2\hat{F}(\lambda)$ twice from zero up to t and using the relations $F(0) = 0$ and $\Theta_t'(0,\lambda) = 0$, we obtain

$$\lambda(\lambda^2 I - A)^{-1}F(t)x = \Theta(t,\lambda)x = \int_0^t S(s,A)\lambda^2\hat{F}(\lambda)x\,ds,\ x \in E, \tag{2.37}$$

and, therefore, assertion (iv) is proved. Assertion (v) is proved analogously.

Remark 2.4.1. If $C(\cdot,A)$ is uniformly continuous, then each C_0-family of multiplicative perturbations $F(\cdot)$ (resp. each C_0-family of additive perturbations) of a C_0-cosine operator function $C(\cdot,A)$ is also uniformly continuous. This follows from formula (iv) (resp. (v)) and Proposition 2.4.1.

Definition 2.4.1. Let $F(\cdot)$ be a C_0-family of multiplicative perturbations for a C_0-cosine operator function $C(\cdot,A)$. The *infinitesimal operator* W_s of the family $F(\cdot)$ is defined as $W_s x = s\text{-}\lim_{h\to 0}\dfrac{2}{h^2}F(h)x$, with a natural domain. The *infinitesimal operator* A_s of the pair $(C(\cdot,A),F(\cdot))$ is defined as $A_s x := s\text{-}\lim_{h\to 0}\dfrac{2}{h^2}(C(h,A) + F(h) - I)x$, with a natural domain. The *infinitesimal operator* W_c of a C_0-family of additive perturbations $G(\cdot)$ and the *infinitesimal operator* A_c of the pair $G(\cdot), C(\cdot,A))$ are defined in the same way as

$$W_c x = s\text{-}\lim_{h\to 0}\frac{2}{h^2}G(h)x \quad\text{and}\quad A_c x := s\text{-}\lim_{h\to 0}\frac{2}{h^2}\Big(C(h,A) + G(h) - I\Big)x,$$

respectively.

Theorem 2.4.1 ([239]). *The operators* W_s *and* A_s *defined above are closed and*

(i) $W_s = \lambda(\lambda^2 - A)\hat{F}(\lambda)$, $\operatorname{Re}\lambda > \omega$;

(ii) $A_s = A(I - \lambda\hat{F}(\lambda)) + \lambda^3\hat{F}(\lambda)$, $\operatorname{Re}\lambda > \omega$;

(iii)

$$A_s = A\Big(I - \frac{2}{t^2}\int_0^t\int_0^\tau F(s)dsd\tau\Big) + \frac{2}{t^2}\Big(\lambda^2\int_0^t\int_0^\tau C(s,A)dsd\tau - (C(t,A) - I)\Big)\lambda\hat{F}(\lambda),$$

where $t \in \mathbb{R}_+$, $\operatorname{Re}\lambda > \omega$.

Proof. Let $A_h = \frac{2}{h^2}\Big(C(h,A) + F(h) - I\Big)$. Assertion (iv) of Proposition 2.4.1 can be rewritten in the form

$$\frac{2F(h)}{h^2}x = \frac{2}{h^2}\int_0^h S(s,A)\lambda^3\hat{F}(\lambda)xds - \frac{2}{h^2}(C(h,A) - I)\lambda\hat{F}(\lambda)x,$$

$$A_h x = 2h^{-2}\int_0^h S(s,A)\lambda^3\hat{F}(\lambda)x\,ds + 2h^{-2}(C(h,A) - I)(I - \lambda\hat{F}(\lambda))x.$$

Since the first term in the right-hand side of each of these relations converges to $\lambda^3\hat{F}(\lambda)x$ as $h \to 0$ by (2.17), we have

$$D(W_s) = D(A\hat{F}(\lambda)) \quad \text{and } W_s x = \lambda(\lambda^2 - A)\hat{F}(\lambda)x \quad \text{for all } x \in D(W_s),$$

and also

$$D(A_s) = D\Big(A(I - \lambda\hat{F}(\lambda))\Big) \text{ and } A_s x = \lambda^3\hat{F}(\lambda)x + A(I - \lambda\hat{F}(\lambda))x \quad \text{for all } x \in D(A_s).$$

Since A is closed and $\hat{F}(\lambda)$ is bounded, it is easy to see that W_s and A_s are closed. Assertions (i) and (ii) are proved.

To prove (iii), we use relation (2.1). For all $x \in E$ and $s \in \overline{\mathbb{R}}_+$, we have

$$\begin{aligned}
\frac{2}{h^2}\Big(C(h,A) + F(h) - I\Big)x &= \frac{2}{h^2}\Big(C(h,A) - I\Big)x + \frac{2}{h^2}\Bigg(F(s+h) - 2C(h,A)F(s) + F(s-h)\Bigg)x \\
&= \frac{2}{h^2}\Big(C(h,A) - I\Big)\Big(I - F(s)\Big)x + \frac{1}{h^2}\Bigg(F(s+h) - 2F(s) + F(s-h)\Bigg)x \\
&= \frac{2}{h^2}(C(h,A) - I)(I - F(s))x + \frac{2}{h^2}C(s,A)F(h)x.
\end{aligned}$$

Integrating twice for any $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
\frac{2}{h^2}(C(h,A) + F(h) - I)x &= \frac{2}{h^2}(C(h,A) - I)\Bigg(I - \frac{2}{t^2}\int_0^t\int_0^\tau F(s)dsd\tau\Bigg)x \\
&+ \frac{2}{t^2}(\lambda^2 I - A)\int_0^t\int_0^\tau C(s,A)dsd\tau(\lambda^2 I - A)^{-1}\frac{2}{h^2}F(h)x.
\end{aligned}$$

Since the last term converges to

$$\frac{2}{t^2}(\lambda^2 - A)\int_0^t\int_0^\tau C(s,A)\,dsd\tau\lambda\hat{F}(\lambda)x$$

as $h \to 0+$ for all $x \in E$ (see Proposition 2.4.1 (iii)), we obtain A_s in the form (iii).

Remark 2.4.2. The definition of the infinitesimal operator via the limit $s\text{-}\lim_{h\to 0+} h^{-1}F(h)$ has no sense. Indeed, in this case, using (2.37), we obtain that such an operator is zero.

Generally speaking, the domains of the operators W_s and A_s are not necessarily dense in E. But under certain conditions on $F(\cdot)$, the operator A_s not only has a dense domain but generates a C_0-cosine operator function. The domains of $D(W_c)$ and $D(A_c)$ always contain the dense set $D(A)$.

Theorem 2.4.2 ([239]). *The infinitesimal operators W_c and A_c have the following properties for* $\operatorname{Re}\lambda > \omega$*:*

(i) $D(A) \subseteq D(W_c)$ *and* $W_c x = \lambda\hat{G}(\lambda)(\lambda^2 - A)x$ *for all* $x \in D(A)$;

(ii) $D(A) \subseteq D(A_c)$, *and for* $x \in D(A)$, *we have*

$$A_c x = Ax + W_c x = \Big(I - \lambda\hat{G}(\lambda)\Big)Ax + \lambda^3\hat{G}(\lambda)x;$$

(iii) $D(A) \subseteq D(A_c)$, *and for all* $x \in D(A)$ *and* $t \in \mathbb{R}_+$

$$A_c x = \Big(I - \frac{2}{t^2}\int_0^t\int_0^\tau G(s)dsd\tau\Big)Ax + \lambda\hat{G}(\lambda)\frac{2}{t^2}\Big(\lambda^2\int_0^t\int_0^\tau C(s,A)dsd\tau - (C(t,A) - I)\Big)x.$$

Moreover, if $G(t)$ is uniformly continuous in t, then A_c is closed, $D(A_c) = D(A)$, and $A_c = (I - \lambda\hat{G}(\lambda))A + \lambda^3\hat{G}(\lambda)$ for large $\operatorname{Re}\lambda$. *If $\hat{G}(\lambda)$ is invertible for a certain λ, then the operator W_c is closed, $D(W_c) = D(A)$, and $W_c = A_c - A = \lambda\hat{G}(\lambda)(\lambda^2 I - A)$.*

Proof. Let $A_h = \frac{2}{h^2}\Big(C(h,A) + G(h) - I\Big)$. By (v) of Proposition 2.4.1, we have

$$\frac{2G(h)}{h^2}x = \lambda^3\hat{G}(\lambda)\frac{2}{h^2}\int_0^h S(s,A)xds - \lambda\hat{G}(\lambda)\frac{2}{h^2}\Big(C(h,A) - I\Big)x,$$

$$A_h x = 2\lambda^3\hat{G}(\lambda)h^{-2}\int_0^h S(s,A)xds + 2(I - \lambda\hat{G}(\lambda))h^{-2}(C(h,A) - I)x.$$

The first identity implies $D(A) \subseteq D(W_c)$ and $W_c x = \lambda\hat{G}(\lambda)(\lambda^2 I - A)x$ for $x \in D(A)$.

The second identity implies $D(A) \subseteq D(A_c)$ and $A_c x = Ax + W_c x = (I - \lambda\hat{G}(\lambda))Ax + \lambda^3\hat{G}(\lambda)x$ for $x \in D(A)$.

The proof of (iii) is similar to that of (iii) in Theorem 2.2.2.

If $\|G(t)\| \to 0$ as $t \to 0$, then $\|\lambda\hat{G}(\lambda)\| \to 0$ as $\lambda \to \infty$ (Proposition 2.4.2 (ii)). Therefore, the operator $I - \lambda\hat{G}(\lambda)$ is invertible for large λ, and we have $D(A_c) \subseteq D(A)$. If $\{x_n\}$ is a sequence in $D(A)$

such that $x_n \to x$ and $(I - \lambda\hat{G}(\lambda))Ax_n \to y$, then $Ax_n \to (I - \lambda\hat{G}(\lambda))^{-1}y$, so that $x \in D(A)$ and $Ax = (I - \lambda\hat{G}(\lambda))^{-1}y$.

Therefore, $(I - \lambda\hat{G}(\lambda))A$ is closed, and hence A_c is also closed. The proofs of the assertions concerning the operator W_c are going in the line as that for A_c. It follows from (2.1) that if $\|F(t)x\| = o(t^2)\,(t \to 0+)$ for all $x \in E$, then $F''(t) = 0$ for all $t \in \overline{\mathbb{R}}_+$, so that $F'(\cdot) \equiv F'(0) = 0$, and then $F(\cdot) \equiv F(0) = 0$.

Similarly, it follows from (2.2) that the condition $\|G(t)x\| = o(t^2)\,(t \to 0+)$ for all $x \in E$ implies $G(\cdot) \equiv 0$. Therefore, the rate of convergence to 0 in the case of a nontrivial multiplicative perturbation family or an additive perturbation family cannot exceed $O(t^2)$ for $t \to 0$.

Proposition 2.4.2 ([239]). *We have the following assertions concerning the rate of convergence to zero*:

(i) *for* $n = 0, 1$, *if* $\|F(t)x\| = o(t^n)$ *as* $t \to 0+$ *for all* $x \in E$, *then* $\|\lambda^n\hat{F}(\lambda)\| = o(1)$ *as* $\lambda \to \infty$ *and* $\|\lambda^{n+1}\hat{F}(\lambda)x\| = o(1)$ *as* $\lambda \to \infty$ *for all* $x \in E$;

(ii) *for* $n = 0, 1$, *if* $\|G(t)x\| = o(t^n)$ *as* $t \to 0+$ *for all* $x \in E$, *then* $\|\lambda^n\hat{G}(\lambda)\| = o(1)$ *as* $\lambda \to \infty$, $\|\lambda^{n+1}\hat{G}(\lambda)x\| = o(1)$ *as* $\lambda \to \infty$ *for all* $x \in E$, *and* $\|\lambda^3\hat{G}(\lambda)x - (A_c - A)x\| = o(\lambda^{-n})$ *for all* $x \in D(A)$;

(iii) *for* $n = 0, 1$, *if* $\|F(t)\| = o(t^n)$ *(resp.* $\|G(t)\| = o(t^n)$*) as* $t \to 0+$, *then* $\|\lambda^{n+1}\hat{F}(\lambda)\| = o(1)$ *(resp.* $\|\lambda^{n+1}\hat{G}(\lambda)\| = o(1)$*) as* $\lambda \to \infty$;

(iv) *for* $n = 1, 2$, *if* $\|F(t)\| = O(t^n)\,(t \to 0+)$, *then* $\|\lambda^{n+1}\hat{F}(\lambda)\| = O(1)$ *as* $\lambda \to \infty$;

(v) *for* $n = 1, 2$, *if* $\|G(t)\| = O(t^n)$ *as* $t \to 0+$, *then* $\|\lambda^{n+1}\hat{G}(\lambda)\| = O(1)$ *as* $\lambda \to \infty$, *and* $\|\lambda^3\hat{G}(\lambda)x - (A_c - A)x\| = O(\lambda^{-n})$ *for all* $x \in D(A)$;

(vi) *if* $\|F(t)\| = O(t^2)$ *as* $t \to 0$, *then* $w^*\text{-}\lim_{\lambda\to\infty} \lambda^3\hat{F}(\lambda)^*x^* = (A_s^* - A^*)x^*$ *for any* $x^* \in D(A^*)$.

Proof. We prove only (ii); the proof of assertions (i), (iii), (iv), and (v) is analogous. For a given $\epsilon > 0$, choose $\delta > 0$ such that $\|G(t)x\| \le \epsilon t^n$ for all $t \in [0, \delta]$. Then

$$\|\lambda^{n+1}\hat{G}(\lambda)x\| \le \lambda^{n+1}\Big(\int_0^\delta + \int_\delta^\infty\Big)e^{-\lambda t}\|G(t)x\|dt$$

$$\le \epsilon\lambda^{n+1}\int_0^\infty e^{-\lambda t}t^n\,dt + \lambda^{n+1}\int_\delta^\infty e^{-\lambda t}Me^{\omega t}dt\|x\| \le \epsilon/n! + M\frac{\lambda^{n+1}}{\lambda - \omega}e^{-(\lambda-\omega)\delta}\|x\|.$$

This implies $\|\lambda^{n+1}\hat{G}(\lambda)x\| = o(1)$ as $\lambda \to \infty$ for $x \in E$. By the uniform boundedness principle, we have $\|\lambda^n\hat{G}(\lambda)\| = o(1)$ as $\lambda \to \infty$. Now, we have from (ii) of Theorem 2.4.2 that $\|\lambda^3\hat{G}(\lambda)x - (A_c - A)x\| = o(\lambda^{-n})$ for all $x \in D(A)$.

Chapter 3

REDUCTION OF THE CAUCHY PROBLEM FOR A SECOND ORDER EQUATION TO THE CAUCHY PROBLEM FOR A SYSTEM OF FIRST ORDER EQUATIONS

As for ordinary differential equations, nth order equations can be reduced to a set of first order equations by using matrix operators. The matrix operator theory is presented in [128] in detail. In the present chapter, we consider only problems of reducing incomplete second order equations to a set of first order equation.

3.1. Kysinski Theorem

In a Banach space E, let us consider the following uniformly well-posed Cauchy problem:

$$u''(t) = Au(t), \quad t \in \mathbb{R}; \qquad u(0) = u^0, \quad u'(0) = u^1. \tag{3.1}$$

Define the matrix operator $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} : E^1 \times E \to E^1 \times E$ acting on an element $(x, y) \in E^1 \times E$ by the formula $\mathcal{A}(x, y) = (y, Ax)$ that is given on the domain $D(\mathcal{A}) = D(A) \times E^1$. In what follows, an element $(x, y) \in E^1 \times E$ in the paragraph formulas will be written as the vector $\binom{x}{y}$.

Theorem 3.1.1 ([179]). *The space E^1 with the norm*

$$\|x\|_{E^1} := \|x\| + \sup_{0 \le t \le 1} \|C'(t, A)x\| \tag{3.2}$$

is a Banach space, and the operator $\mathcal{A}$ generates the following C_0-groups of operators on the Banach space $E^1 \times E$:

$$\exp(t\mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} C(t, A) & S(t, A) \\ AS(t, A) & C(t, A) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C(t, A)x + S(t, A)y \\ AS(t, A)x + C(t, A)y \end{pmatrix}, \quad t \in \mathbb{R}.$$

Proposition 3.1.1 ([287]). *Let a C_0-cosine operator function $C(\cdot, A)$ be given. Then E^1 coincides with the closure of $D(A)$ in the norm*

$$\|x\|^* := \|x\| + \sup_{z > \omega, n \in \mathbb{N}} \frac{1}{n!}(z - \omega)^{n+1} \left\| \frac{d^n}{dz^n} A(z^2 I - A)^{-1} x \right\|. \tag{3.3}$$

Proposition 3.1.2 ([179]). *The resolvent of the operator $\mathcal{A}$ has the form*

$$(\lambda I - \mathcal{A})^{-1} = \begin{pmatrix} \lambda(\lambda^2 I - A)^{-1} & (\lambda^2 I - A)^{-1} \\ A(\lambda^2 I - A)^{-1} & \lambda(\lambda^2 I - A)^{-1} \end{pmatrix} \quad \textit{for} \quad \lambda^2 \in \rho(A). \tag{3.4}$$

Proposition 3.1.3 ([179]). *Let $u(\cdot)$ be a solution of problem* (3.1), *and let $v(t) := u'(t), t \in \mathbb{R}$. Then the vector $\begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix}$ is a solution of the following uniformly well-posed Cauchy problem in the Banach space $E^1 \times E$:*

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}(t), \quad t \in \mathbb{R}; \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}. \tag{3.5}$$

Proposition 3.1.4 ([179]). *Let a certain Banach space $\tilde{E}^1$ be continuously and densely embedded into the Banach space E, and, moreover, let $D(\tilde{A}) \subseteq \tilde{E}^1$ for a certain operator $\tilde{A} \in L(E)$. Then if in the space $\tilde{E}^1 \times E$, the Cauchy problem*

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \begin{pmatrix} 0 & I \\ \tilde{A} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}(t) \equiv \tilde{\mathcal{A}}\begin{pmatrix} u \\ v \end{pmatrix}(t), \quad t \in \mathbb{R}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}, \tag{3.6}$$

is uniformly well posed and $\rho(\tilde{A}) \neq \varnothing$, we have $\tilde{A} \in \mathcal{C}(M, \omega)$.

Proposition 3.1.5 ([179]). *Under the conditions of Proposition* 3.1.4, *the C_0-semigroup corresponding to problem* (3.6) *on the space $\tilde{E}^1 \times E$ is represented in the form*

$$\exp(t\tilde{\mathcal{A}}) := \begin{pmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{pmatrix}, \quad t \in \overline{\mathbb{R}_+}, \tag{3.7}$$

where the family $G_{22}(\cdot)$ is a C_0-cosine operator function $C(\cdot, A)$ and coincides with $G_{11}(\cdot)$ on $\tilde{E}^1$.

Proposition 3.1.6 ([179]). *Under the conditions of Proposition* 3.1.4, *the following relations hold for $x \in E$ and $y \in \tilde{E}^1$:*

$$G_{12}(t)x = S(t, A)x \quad \textit{and} \quad G_{21}(t)y = C'(t, A)y = AS(t, A)y, \quad t \in \mathbb{R}.$$

Proposition 3.1.7 ([179]). *The spaces $\tilde{E}^1$ and E^1 coincide with accuracy up to a norm equivalence.*

3.2. Conditions (K) and (F)

However, we note that to study problem (3.1) by reducing it to system (3.5) is very inconvenient, since the space E^1 is defined either through the C_0-cosine operator function $C(\cdot, A)$ or through infinitely many powers of the resolvent. As a rule, we have only the information about the operator A. Therefore, certain additional conditions that allows us to reduce problem (3.1) to a system without use of the space E^1 are of interest.

Proposition 3.2.1 ([274]). *Let the space E be Hilbert, and let the operator A be self-adjoint and negative-definite. Then $A \in \mathcal{C}(M, \omega)$, and the corresponding space E^1 coincides with $\mathcal{D}((-A)^{1/2})$.*

Let the uniformly well-posed problem (3.1) have the form

$$u''(t) = \mathfrak{B}^2 u(t); \quad t \in \mathbb{R}, \quad u(0) = u^0, \quad u'(0) = u^1, \tag{3.8}$$

where $\mathfrak{B} \in \mathcal{C}(E)$.

Definition 3.2.1. We say that a solution $u(\cdot)$ of problem (3.8) satisfies Condition (K) if $u'(\cdot) \in C\Big([0,T]; \mathcal{D}(\mathfrak{B})\Big)$.

Proposition 3.2.2 ([47])**.** *Problem* (3.8) *has a unique solution satisfying Condition* (K) *iff the following Cauchy problem is uniformly well posed on the space* $E \times E$:

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}(t), \quad t \in \mathbb{R}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{3.9}$$

An analog of Condition (K), which allows us to simplify the study of problem (3.1) by using C_0-semigroups, is the following Condition (F).

Definition 3.2.2. A C_0-cosine operator function $C(\cdot, A)$ satisfies Condition (F) if the following conditions hold:

(i) there exists $\mathfrak{B} \in \mathcal{C}(E)$ such that $\mathfrak{B}^2 = A$, and $\mathfrak{B}$ commutes with any operator from $B(E)$ commuting with A;

(ii) $S(t, A)$ maps E into $D(\mathfrak{B})$ for any $t \in \mathbb{R}$;

(iii) the function $\mathfrak{B}S(t, A)x$ is continuous in $t \in \mathbb{R}$ for any fixed $x \in E$.

Proposition 3.2.3 ([135])**.** *Under Condition (F), for each $t \in \mathbb{R}$, we have $\mathfrak{B}S(t, A) \in B(E)$ and $D(\mathfrak{B}) \subseteq E^1$.*

Proposition 3.2.4 ([135])**.** *There exist a Banach space E and a C_0-cosine operator function $C(\cdot, A)$ (even uniformly bounded) such that Condition (F) does not hold.*

Proposition 3.2.5 ([135])**.** *Via the shift $A_b := A - b^2 I$ for $b > \omega_c(A)$, we can always construct operators A_b and $\mathfrak{B}_b$ such that $\mathfrak{B}_b^2 = A_b$ and $\mathfrak{B}_b$ commutes with any operator from $B(E)$ commuting with A_b.*

Proposition 3.2.6 ([134])**.** *The operator $\mathfrak{B}_b$ in Proposition* 3.2.5 *can be constructed, e.g., as follows*:

$$\mathfrak{B}_b x := \frac{-i}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda I - A_b)^{-1} (-A_b x) d\lambda.$$

Theorem 3.2.1 ([272]). *Let A and $\mathfrak{B}$ be operators satisfying condition* (i) *in Definition* 3.2.2, *and let* $0 \in \rho(\mathfrak{B})$. *The following conditions are equivalent*:

(i) *the C_0-cosine operator function $C(\cdot, A)$ satisfies Condition* (F);

(ii) *the operator $\mathfrak{B}$ generates a C_0-semigroup* $\exp(\cdot\mathfrak{B})$ *on E*;

(iii) *the operator* $\begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix}$ *with the domain $D(A) \times D(\mathfrak{B})$ generates a C_0-group on $E \times E$*;

(iv) *the operator* $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ *with the domain $D(A) \times D(\mathfrak{B})$ generates a C_0-group* $\exp(\cdot\mathcal{A})$ *on $\mathcal{D}(\mathfrak{B}) \times E$, where $\mathcal{D}(\mathfrak{B})$ is the Banach space of elements $D(\mathfrak{B})$ endowed with the graph norm*;

(v) *the embedding $D(\mathfrak{B}) \subseteq E^1$ holds*;

(vi) $D(\mathfrak{B}) = E^1$.

Proposition 3.2.7 ([104]). *Let $A \in \mathcal{C}(M, 0)$, and let E be a UMD space. Then Condition* (F) *holds.*

Proposition 3.2.8 ([288]). *The following condition is equivalent to conditions* (i)–(vi) *of Theorem* 3.2.1: *$D(\mathfrak{B})$ is dense in E, and there exist constants $M > 0$ and $\omega \geq 0$ such that $\lambda^2 \in \rho(A)$ for any $\lambda > \omega$, the operator functions $\lambda(\lambda^2 I - A)^{-1}$ and $\mathfrak{B}(\lambda^2 I - A)^{-1}$ are strongly infinitely many times differentiable for $\lambda > \omega$, and the following estimates hold for any $m \in \mathbb{N}_0$*:

$$\left\| \frac{(\lambda-\omega)^{m+1}}{m!} \left(\frac{d}{d\lambda}\right)^m \left(\lambda(\lambda^2 I - A)^{-1}\right) \right\| \leq M,$$

$$\left\| \frac{(\lambda-\omega)^{m+1}}{m!} \left(\frac{d}{d\lambda}\right)^m \left(\mathfrak{B}(\lambda^2 I - A)^{-1}\right) \right\| \leq M.$$

Proposition 3.2.9 ([272]). *Under the conditions of Theorem* 3.2.1, *for $t \in \mathbb{R}$, we have*

$$\text{(i)} \quad \exp(t\mathfrak{B}) = C(t, A) + \mathfrak{B}S(t, A), \;\; C(t, A) = \Big(\exp(t\mathfrak{B}) + \exp(-t\mathfrak{B})\Big)/2;$$

$$\text{(ii)} \quad \exp(t\mathcal{A}) = \begin{pmatrix} \mathfrak{B}^{-1} & 0 \\ 0 & I \end{pmatrix} \exp\left(t \begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix}\right) \begin{pmatrix} \mathfrak{B} & 0 \\ 0 & I \end{pmatrix};$$

$$\text{(iii)} \quad \exp(t\mathcal{A}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C(t, A)x + S(t, A)y \\ AS(t, A)x + C(t, A)y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(\mathfrak{B}) \times E.$$

In applications, there often arises the following system of the special form:

$$\begin{cases} u'(t) = -A_0 u(t) + Bv(t), & u(0) = x, \\ v'(t) = Cu(t) - A_1 v(t), & v(0) = y, \end{cases}$$

on the space $\mathcal{H} = H_0 \times H_1$ with linear operators

$$A_0 : D(A_0) \subseteq H_0 \to H_0, \quad A_1 : D(A_1) \subseteq H_1 \to H_1,$$

$$B : D(B) \subseteq H_1 \to H_0, \quad C : D(C) \subseteq H_0 \to H_1.$$

The corresponding matrix operator $\mathcal{A}$ is defined as follows:

$$\mathcal{A} = \begin{pmatrix} -A_0 & B \\ C & -A_1 \end{pmatrix}$$

on $\mathcal{H}$ with

$$D(\mathcal{A}) = \Big(D(A_0) \cap D(C)\Big) \times \Big(D(A_1) \cap D(B)\Big).$$

Theorem 3.2.2 ([195]). *Let* $\exp(t, -A_0)$ *and* $\exp(t, -A_1)$ *be contractive* C_0*-semigroups on* H_0 *and* H_1*, respectively, and let* B *and* C *be closed, and, moreover,* $\overline{D(A_0) \cap D(C)} = H_0$ *and* $\overline{D(A_1) \cap D(B)} = H_1$*. Also, let* $\operatorname{Re}\{\langle A_0 x, x\rangle + \langle A_1 y, y\rangle - \langle By, x\rangle - \langle Cx, y\rangle\} \geq 0$ *for any* $x \in D(A_0) \cap D(C)$ *and* $y \in D(A_1) \cap D(B)$*.*

Then the following conditions are equivalent:

(i) $\mathcal{A}$ *generates a contractive* C_0*-semigroup on* $\mathcal{H}$*;*

(ii) *for any* $\lambda > 0$*, we have*

$$(\lambda I + A_0 - B(\lambda I + A_1)^{-1} C)^{-1} \in B(H_0), \quad (\lambda I + A_1 - C(\lambda I + A_0)^{-1} B)^{-1} \in B(H_1);$$

(iii) *assertions* (ii) *hold for a certain* $\lambda > 0$*;*

(iv) *for any* $\lambda > 0$*, the operators* $-(A_0 - B(\lambda I + A_1)^{-1} C)$ *and* $-(A_1 - C(\lambda I + A_0)^{-1} B)$ *generate contractive* C_0*-semigroups on* H_0 *and* H_1*, respectively;*

(v) *assertions* (iv) *hold for a certain* $\lambda > 0$*.*

Also, in [195], the conditions under which the operator $\mathcal{A}$ generates an exponentially stable, differentiable, and analytic C_0-semigroup belonging to the Gevrey class with $\delta > 0$ were obtained.

Chapter 4

INTERPOLATION

The interpolation theory considerably increases the total volume of results in the theory of partial differential equations. We will mostly interest in two global directions: applications to coercive inequalities, which often do not hold in the traditional spaces, and applications to the rate of convergence of approximative methods depending on the smoothness of initial data (see the first article in this volume).

Let X and Y be two complex Banach spaces continuously embedded in a Hausdorff topological vector space E, i.e., $X \hookrightarrow E$ and $Y \hookrightarrow E$. Such Banach spaces X and Y are called an interpolation pair, which is denoted by $\{X, Y\}$.

Proposition 4.1.1 ([9]). *Let $\{X, Y\}$ be an interpolation pair. Then $X+Y$ and $X \cap Y$ are Banach spaces with the norms*

$$\|x\|_{X\cap Y} = \max(\|x\|_X, \|x\|_Y), \quad \|x\|_{X+Y} = \inf_{\substack{x=x_0+x_1 \\ x_0\in X,\, x_1\in Y}} \{\|x_0\|_X, \|x_1\|_Y\},$$

respectively.

Obviously, if $Y \hookrightarrow X$, then $X \cap Y = Y$ and $X + Y = X$. In such a case, it is natural to set $E = X$, which usually holds in applications.

Definition 4.1.1. For any $t \in \mathbb{R}_+$ and an interpolation pair $\{X_0, X_1\}$, we define the so-called *Peetre K-functional*

$$K(t, x; X_0, X_1) = \inf_{\substack{x=x_0+x_1, \\ x_0\in X_0, x_1\in X_1}} (\|x_0\|_{X_0} + t\,\|x_1\|_{X_1})$$

for any $x \in X_0 + X_1$.

Sometimes, one merely writes $K(t, x)$ if the choice of the spaces X_0 and X_1 is clear.

Definition 4.1.2. The *interpolation space* $(X_0, X_1)_{\theta,q}$, $0 \le \theta \le 1$, $1 \le q < \infty$, constructed according to an interpolation pair $\{X_0, X_1\}$ by using the K-method, is the space of all elements $x \in X_0 + X_1$ for which the following norm is finite:

$$\|x\|_{(X_0,X_1)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t,x)\right)^q dt \right)^{\frac{1}{q}}.$$

In the case $q = \infty$, instead of $(X_0, X_1)_{\theta,\infty}$, one usually writes $(X_0, X_1)_\theta$ and defines the norm as

$$\|x\|_\theta = \sup_{0<t<\infty} t^{-\theta} K(t, x).$$

The interpolation space with $q = \infty$ is of a specific interest in considering approximations of C_0-semigroups of operators and C_0-cosine operator functions by using the Favard classes.

Along with the K-functional, it is possible to use other constructions for constructing interpolation spaces. For more detail, see, e.g., [9, 73].

Definition 4.1.3. We say that a space $\mathcal{E} \in K_\theta(X_0, X_1)$, if it is continuously embedded in $(X_0, X_1)_\theta$, i.e., $K(t, x) \le ct^\theta \|x\|_{\mathcal{E}}$ for any $x \in \mathcal{E}$.

In connection with Definition 4.1.3, for an interpolation pair $\{X_0, X_1\}$, it is useful to set

$$\mathcal{J}_j(X_0, X_1) \cap K_j(X_0, X_1) = \{X_j\}, \quad j = 0, 1.$$

Definition 4.1.4. A Banach space $[X_0, X_1]_\theta$ constructed by using the complex interpolation method is called the *interpolation space* corresponding to an interpolation pair $\{X_0, X_1\}$.

Definition 4.1.5. Let Ω be an open set in $\mathbb{R}^d$, $m \in \mathbb{N}_0$, and let $1 \le q, p \le \infty$. Let $\sigma = m + \theta$, where $0 < \theta \le 1$.

We set $\Delta_y f(x) := f(x+y) - f(x)$, $\Delta_y^2 f(x) := f(x+2y) - 2f(x+y) + f(x)$, and $\Omega_{k,y} := \bigcap\limits_{j=0}^{k} (\Omega - jy) = \{x : x + jy \in \Omega \text{ for } j = \overline{0, k}\}$.

The *Besov space* $B^\sigma_{p,q}(\Omega, E)$ is defined as the space of all functions f from $W^m_p(\Omega; E)$, for which the seminorm

$$|f|_{B^\sigma_{p,q}\left(\Omega;E\right)} := \sum_{|\alpha|=m} \| |y|^{-\theta} \{ \|\Delta_y^k \partial_x^\alpha f(x)\|_{L^p(\Omega_{k,y};E)} \} \|_{L^q_*(\mathbb{R}^d)}$$

is finite for $k = 1$ or $k = 2$ when $0 < \theta < 1$ or $\theta = 1$, respectively.

The *norm of the Besov space* is defined as follows:

$$\|f\|_{B^\sigma_{p,q}\left(\Omega;E\right)} := \|f\|_{L^p\left(\Omega;E\right)} + |f|_{B^\sigma_{p,q}\left(\Omega;E\right)}.$$

Here $L^p_*(\Omega)$ is an $L^p(\Omega)$ space with the measure $|x|^{-d}\, dx$, $\Omega \subseteq \mathbb{R}^d$.

Theorem 4.1.1 ([219])**.** *Let $a, b < \infty$. Then*

(i) *for $\sigma, \tau \in \mathbb{R}$, $1 \le p_1 \le p_2 \le \infty$, in the case $\sigma - \dfrac{1}{p_1} > \tau - \dfrac{1}{p_2}$ or $\sigma - \dfrac{1}{p_1} = \tau - \dfrac{1}{p_2}$ and $q_1 \le q_2$, we have $B^\sigma_{p_1,q_1}((a,b), E) \hookrightarrow B^\tau_{p_2,q_2}((a,b), E)$;*

(ii) *$B^m_{p,1}((a,b), E) \subset W^m_p((a,b), E) \subseteq B^m_{p,\infty}\Big((a,b), E\Big)$ for any $m \in \mathbb{N}$.*

In particular, $B^m_{\infty,1}\Big((a,b), E\Big) \subset C^m\Big((a,b), E\Big)$.

4.2. Interpolation in the C_0-Semigroup Theory

Recall that by $\mathcal{D}(A^m)$ we denote the Banach space of elements $x \in D(A^m)$ endowed with the norm $\|x\|_{\mathcal{D}(A^m)} = \|x\| + \|A^m x\|$.

Theorem 4.2.1 ([73])**.** *Let $m \in \mathbb{N}$, $0 < \theta < 1$, $1 \le p < \infty$ and $k, l \in \mathbb{Z}$ with $0 \le k < s = \theta m$, $l > s - k$. Then*

(i) *for $A \in \mathcal{G}(M, \omega)$ and $0 < \delta < \infty$,*

$$(E, \mathcal{D}(A^m))_{\theta,p} = \Big\{ x \in E : \|x\|^{(k,l,\delta)}_{(E,\mathcal{D}(A^m))_{\theta,p}} < \infty \Big\},$$

where

$$\|x\|^{(k,l,\delta)}_{(E,\mathcal{D}(A^m))_{\theta,p}} = \|x\|_E + \left(\int_0^\delta \|t^{-(s-k)}(\exp(tA) - I)^l A^k x\|_E^p \frac{dt}{t} \right)^{\frac{1}{p}},$$

and all these norms are equivalent to the norm $\|\cdot\|_{(E,\mathcal{D}(A^m))_{\theta,p}}$;

(ii) *if* $\omega < 0$, *then* $\delta = \infty$ *is an admissible value in the definition of the norm.*

Definition 4.2.1. An operator $A \in \mathcal{C}(E)$ is said to be *positive* if $(-\infty, 0] \subseteq \rho(A)$ and there exists a number $C > 0$ such that

$$\|(A - \lambda I)^{-1}\| \leq \frac{C}{1 + |\lambda|} \quad \text{for} \quad \lambda \in (-\infty, 0].$$

Note that in the case $A \in \mathcal{G}(M, \omega)$ with $\omega < 0$, the operator $-A$ is positive.

Theorem 4.2.2 ([73])**.** *Let $-A$ be positive, and let $m \in \mathbb{N}$, $0 < \theta < 1$, $1 \leq p \leq \infty$. Then*

$$(E, \mathcal{D}(A^m))_{\theta,p} = \left\{ x \in E : \|x\|_* = \Big(\int_0^\infty (t^{\theta m} \|A^m (tI + A)^{-m} x\|_E)^p \frac{dt}{t} \Big)^{\frac{1}{p}} < \infty \right\};$$

moreover, the norm $\|\cdot\|_*$ *is equivalent to the norm of the space* $(E, \mathcal{D}(A^m))_{\theta,p}$.

Theorem 4.2.3 ([73])**.** *Let $-A$ be a positive operator. Then*

(i) *if* $j, m \in \mathbb{N}$ *and* $1 \leq j \leq m$, *then*

$$(E, \mathcal{D}(A^m))_{j/m,1} \subseteq \mathcal{D}(A^j) \subseteq (E, \mathcal{D}(A^m))_{j/m,\infty};$$

(ii) *if* $m \in \mathbb{N}$, $0 < \theta < 1$, $1 \leq p \leq \infty$, *and* $k, l \in \mathbb{Z}$, $0 \leq k < s = m\theta$, $l > s - k$, *then*

$$(E, \mathcal{D}(A^m))_{\theta,p} = \left\{ x \in E : \left(\int_0^\infty \left(t^{s-k} \|A^l (tI + A)^{-l} A^k x\|_E \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \right\};$$

(iii) *if* $A \in \mathcal{H}(M, \omega)$ *with* $\omega < 0$, *then*

$$(E, \mathcal{D}(A^m))_{\theta,p} = \left\{ x \in E : \|x\|_{**} = \left(\int_0^\infty \left(t^{m - \theta m} \|A^m \exp(tA) x\|_E \right)^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \right\},$$

where $\|\cdot\|_{**}$ *is a norm equivalent to the norm of the space* $(E, \mathcal{D}(A^m))_{\theta,p}$.

Proposition 4.2.1 ([73])**.** *Let A be a positive operator, and let $\sigma \in \mathbb{R}_+$, $k, m \in \mathbb{Z}$, $k \geq 0$, $0 < \sigma < m$. Then for complex numbers $-k < \operatorname{Re} z \leq \sigma - k$ and $x \in (E, \mathcal{D}(A^m))_{\sigma/m,p}$, the integral*

$$A_\sigma^z x = \frac{\Gamma(m)}{\Gamma(z + m)\Gamma(m - k - z)} \int_0^\infty \lambda^{z+k-1} A^{m-k} (A + \lambda I)^{-m} x \, d\lambda$$

where $\Gamma(m) := \int\limits_0^\infty e^{-mt} t^{m-1} \, dt$ *is the gamma-function, converges. The operator A_σ^z is closable and is independent of σ.*

Definition 4.2.2. Let A be a positive operator, and let $z \in \mathbb{C}$. The *fractional power* A^z of the operator A is defined as the closure of the operator A^z_σ.

Theorem 4.2.4 ([46])**.** *Let A be a positive operator. Then*

(i) *if $m \in \mathbb{N}$, $\operatorname{Re}\alpha, \operatorname{Re}\beta < m$, then*

$$A^\alpha A^\beta x = A^\beta A^\alpha x \quad \text{for } x \in D(A^{2m});$$

(ii) *if $\operatorname{Re}\alpha < 0$, then A^α is a continuous operator and $A^{-\alpha}A^\alpha = I$;*

(iii) *if $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$, then $A^\alpha A^\beta = A^{\alpha+\beta}$;*

(iv) *if $m \in \mathbb{N}$ and $0 < \operatorname{Re}\alpha < m$, then*

$$(E, \mathcal{D}(A^m))_{\frac{\operatorname{Re}\alpha}{m},1} \subseteq \mathcal{D}(A^\alpha) \subseteq (E, \mathcal{D}(A^m))_{\frac{\operatorname{Re}\alpha}{m},\infty};$$

(v) *if $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta < \infty$ and $1 \le p \le \infty, 0 < \theta < 1$, then*

$$(E, \mathcal{D}(A^\alpha))_{\theta,p} = (E, \mathcal{D}(A^\beta))_{\frac{\operatorname{Re}\alpha}{\operatorname{Re}\beta}\theta,p}.$$

Proposition 4.2.2 ([73])**.** *Let A be a positive operator, and let there exist constants ε and C such that A^{it} are operators uniformly bounded near zero, i.e., $\|A^{it}\| \le C$ for $-\varepsilon \le t \le \varepsilon$. If $0 \le \operatorname{Re}\alpha < \operatorname{Re}\beta < \infty$ and $0 < \theta < 1$, then*

$$\left[\mathcal{D}(A^\alpha), \mathcal{D}(A^\beta)\right]_\theta = \mathcal{D}\left(A^{\alpha(1-\theta)+\beta\theta}\right).$$

Proposition 4.2.3 ([98])**.** *Let $A \in \mathcal{G}(M, 0)$, and let $0 < \alpha < 1$. Then the following conditions are equivalent for $x \in E$:*

(i) *$x \in \mathcal{D}((-A)^\alpha)$;*

(ii) *there exists*

$$s\text{-}\lim_{\varepsilon\to 0} \frac{1}{\Gamma(-\alpha)} \int_\varepsilon^\infty t^{-\alpha-1}(\exp(tA) - I)x\, dt.$$

Proposition 4.2.4 ([98])**.** *Let $\tau > 0$, $A \in \mathcal{G}(M, 0)$, and let*

$$U_\beta(\tau)x := \int_0^\tau (\tau - s)^{\beta-1} \exp(sA)x\, ds$$

for $0 < \alpha < \beta \le 1$, $x \in E$. Then $U_\beta(\tau)x \in \mathcal{D}((-A)^\alpha)$.

Proposition 4.2.5 ([98])**.** *Let $A \in \mathcal{G}(M, 0)$ be a normal operator on a Hilbert space $E = H$. Then for the operator function $C^\alpha_t[\exp(\cdot A)]$, we have $C^\alpha_t[\exp(\cdot A)]E \subseteq \mathcal{D}((-A)^\alpha)$ for $0 < \alpha < 1$, and the operator $(-A)^\alpha C^\alpha_t[\exp(\cdot A)]$ is strongly continuous.*

Theorem 4.2.5 (reiteration theorem, [73]). *Let A be a positive operator satisfying the conditions of Proposition* 4.2.2, *and let* $\operatorname{Re}\alpha > 0$. *Then for* $1 \le p < \infty$, $0 < \theta_0 < \theta_1 < 1$, *and* $0 < \lambda < 1$,

$$\Big[(E, \mathcal{D}(A^\alpha)_{\theta_0,p}, (E, \mathcal{D}(A^\alpha)_{\theta_1,p}\Big]_\lambda = \Big(E, \mathcal{D}(A^\alpha)\Big)_{(1-\lambda)\theta_0+\lambda\theta_1,p}.$$

As was already noted, if $A \in \mathcal{H}(\omega, \beta)$ with $\omega \le 0$, then the operator $-A$ is positive. At the same time, the construction of fractional powers is simplified in this case. The location of the spectrum of the operator $A \in \mathcal{H}(\omega, \beta)$ is as follows:

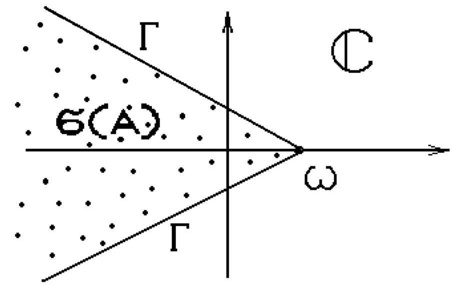

Fig. 1

and $\|(\lambda I - A)^{-1}\| \le \dfrac{M}{|\lambda - \omega|}$ for $\lambda \in \Gamma$. Assume that $\omega = 0$. Then we can set

$$(-A)^{-\alpha} = \frac{1}{2\pi i}\int_\Gamma \lambda^{-\alpha}(\lambda I + A)^{-1}\, d\lambda,\ 0 < \alpha < \infty,$$

where the contour Γ in the integral is going around upward; see Fig. 1.

The operators $(-A)^{-\alpha}$ are bounded, and for integer $\alpha = m \in \mathbb{N}$, we have $(-A)^\alpha = (-A)^{-n}$. Moreover, the operators $(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-(\alpha+\beta)}$ form a semigroup, $\|(-A)^{-\alpha}\| \le \text{const}$, $0 \le \operatorname{Re}\alpha \le 1$, and this semigroup is strongly continuous at zero, i.e., $(-A)^{-\alpha}x \to x$ as $\alpha \to 0$ for any $x \in E$. Complex powers are defined by the formula

$$(-A)^z = \frac{1}{2\pi i}\int_\Gamma \lambda^{-z}(\lambda I + A)^{-1}\, d\lambda, \quad \operatorname{Re} z < 0. \tag{4.1}$$

Proposition 4.2.6 ([73]). *Let $A \in \mathcal{H}(\omega, \beta)$. Then $\{(-A)^z\}_{\operatorname{Re} z \le 0}$ is a C_0-semigroup analytic in the open left half-plane. As the inverses to bounded operators, the operators $(-A)^\alpha$, $0 < \alpha < \infty$, are closed and, moreover, $\overline{D((-A)^\alpha)} = E$.*

Proposition 4.2.7 ([47] (momentum inequality)). *Let A be positive. Then for any $\alpha < \beta < \gamma$, we have*

$$\|A^\beta x\| \le C(\alpha, \beta, \gamma)\|A^\gamma x\|^{\frac{\beta-\alpha}{\gamma-\alpha}} \cdot \|A^\alpha x\|^{\frac{\gamma-\beta}{\gamma-\alpha}} \quad \textit{for} \quad x \in \mathcal{D}(A^\gamma).$$

In [90, 211], fractional powers of a positive operator A are defined as follows:

if A is bounded, then $A^\alpha := \dfrac{\sin(\alpha\pi)}{\pi} \int\limits_0^\infty \mu^{\alpha-1}(\mu I + A)^{-1}x\, d\mu$, $0 < \operatorname{Re}\alpha < 1$;

if A is unbounded and $0 \in \rho(A)$, then $A^\alpha := [(A^{-1})^\alpha]^{-1}$;

if A is unbounded and $0 \in \sigma(A)$, then $A^\alpha x := \lim\limits_{\varepsilon\to 0+} (A + \varepsilon I)^\alpha x$ on those x at which the limit exists.

With such a definition, it is easy to see that $A^1 = A$, and the case where $\overline{\mathcal{D}(A)} \neq E$ and $0 \in \sigma(A)$ turns out to be appropriate. As was shown in [211], it is easy to prove the relations $A^\alpha A^\beta = A^\beta A^\alpha = A^{\alpha+\beta}$, $(A^\alpha)^\beta = A^{\alpha\beta}$, and the integral expansions for the expression $A^\alpha x - \sum\limits_{p=0}^{n} C_p^\alpha (-1)^p \varepsilon^p (A+\varepsilon I)^{\alpha-p} x$, where C_p^α are binomial coefficients. Moreover, the example showing that $\mathcal{D}(A^{s+it_1}) \setminus \mathcal{D}(A^{s+it_2}) \neq \varnothing$ for any $s > 0$ and $t_1 \neq t_2$ was presented in this work.

Proposition 4.2.8 ([253])**.** *The following Landau inequalities hold for $A \in \mathcal{G}(M, 0)$:*

$$\|A^2x\|^3 \le 3\|x\|\|A^3x\|^2,\ \ \|A^2x\|^3 \le \frac{72}{25}\|x\|\|A^3x\|^2,\ \ \|Ax\|^3 \le \frac{81}{40}\|x\|^2\|A^3x\|,$$
$$\|A^3x\|^4 \le C\|x\|\|A^4x\|^3,\ \ \|A^2x\|^4 \le C\|x\|^2\|A^4x\|^2,\ \ \|Ax\|^4 \le C\|x\|^3\|A^4x\|.$$

Theorem 4.2.6 ([47])**.** *Let A be positive. Then the operators $-A^\alpha$ generate analytic C_0-semigroups for $\alpha \le \frac{1}{2}$.*

Theorem 4.2.7 ([47])**.** *Let $A \in \mathcal{H}(\omega, \beta)$ with $\omega < 0$. Then the operator $-(-A)^\alpha$ is a generator of an analytic C_0-semigroup for any $0 \le \alpha \le 1$.*

Theorem 4.2.8 ([126])**.** *Let $\alpha > 0$, and let $A \in \mathcal{H}(\omega, \pi/2)$; moreover, let*

$$\|\exp(zA)\| \le M \left(\frac{|z|}{\operatorname{Re} z}\right)^\alpha \quad \text{for} \quad \operatorname{Re} z > 0.$$

Then the operator $-(-A)^{1/2}$ generates an analytic C_0-semigroup analytic in the right half-plane and

$$\|\exp(-z(-A)^{1/2})\| \le M' \left(\frac{|z|}{\operatorname{Re} z}\right)^{\alpha+\frac{1}{2}} \quad \text{for} \quad \operatorname{Re} z > 0.$$

Moreover, A generates a β times integrated cosine operator function for $\beta > \alpha + \frac{1}{2}$.

Proposition 4.2.9 ([65]). *Let $A \in \mathcal{G}(M,0)$. Then $-(-A)^{1/2}$ generates an analytic C_0-semigroup, and the following representations hold for $t > 0$:*

$$\exp(-t(-A)^{1/2}) = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{4s}} \exp(sA) \frac{ds}{s^{3/2}},$$

$$\exp(-t(-A)^{1/2}) = \frac{t^{1/2}}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{t}{4s}} \exp(tsA) \frac{ds}{s^{3/2}},$$

$$\exp(-t(-A)^{1/2}) = \frac{t^{1/2}}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{ts}{4}} \exp(tA/s) \frac{ds}{s^{1/2}}.$$

Imaginary powers of an operator $-A \in \mathcal{H}(\omega, \beta)$ with the property $0 \in \rho(A)$ can be defined, e.g., as

$$(A)^{is} = g_s(A)(A+I)^2 A^{-1}, \tag{4.2}$$

where $g_s(\lambda) = \lambda^{is} \dfrac{\lambda}{(1+\lambda)^2}$ and $g_s(A) = \dfrac{1}{2\pi i} \int\limits_{\Gamma} g_s(\lambda)(\lambda I - A)^{-1}\, d\lambda$ (see [188]).

4.3. Interpolation in the Theory of C_0-Cosine Operator Functions

As is known, an operator $A \in \mathcal{C}(M,0)$ also defines an analytic C_0-semigroup, and, therefore, following the previous section, we can define its fractional powers A^z. However, we present certain concrete relations that take into account the specific character of a cosine operator functions.

So, by (2.12), we can follow the previous section, and expressing the resolvent through the cosine operator function (see (2.32)), for $b > \omega_c(A)$ we obtain (see [131])

$$(b^2 I - A)^{-\alpha} x = \frac{2^{3/2-\alpha} b^{1/2-\alpha}}{\sqrt{\pi}\Gamma(\alpha)} \int_0^\infty s^{\alpha-1/2} K_{\alpha-1/2}(bs) C(s,A) x\, ds \tag{4.3}$$

for $\alpha > 0$, where K_ν is the Mcdonald's function, which is represented through the Bessel function $I_\nu(t)$ as follows:

$$K_\nu(t) = \frac{\pi}{2} \frac{I_{-\nu}(t) - I_\nu(t)}{\sin(\pi\nu)} \quad \text{for} \quad \nu \neq \pm\pi, \pm 2\pi, \ldots.$$

Let $A \in \mathcal{C}(M,0)$. Then for $k \in \mathbb{N}$ and $k-1 < \alpha < k$, we have the following relation useful in the interpolation theory:

$$(-A)^\alpha x = \frac{1}{C_{\alpha,k}} \int_0^\infty t^{-2\alpha} (C(t,A) - I)^k x \frac{dt}{t}, \quad x \in \mathcal{D}(A^k),$$

where $C_{\alpha,k} = \int\limits_0^\infty t^{-2\alpha} (\cos(t) - 1)^k \dfrac{dt}{t}$.

Proposition 4.3.1 ([166]). *Let $r \in \mathbb{N}, A \in \mathcal{C}(M,0)$, and let $0 < \alpha < r$. An element $\overline{x} \in \mathcal{D}((-A)^\alpha)$ iff there exists the limit*

$$s\text{-}\lim_{\varepsilon \to 0+} \frac{1}{C_{\alpha,r}} \int_\varepsilon^\infty t^{-2\alpha} (C(t,A) - I)^r \overline{x} \frac{dt}{t};$$

in this case, this limit is $-(-A)^\alpha \overline{x}$.

Proposition 4.3.2 ([166]). *Let $r \in \mathbb{N}, A \in \mathcal{C}(M,0)$, and let $0 < \alpha < r$. An element $\overline{x}^* \in \mathcal{D}((-A)^*)$ belongs to $\mathcal{D}((-A)^{\odot})^{\alpha})$ iff $\overline{x}^* \in E^{\odot}$ and there exists the limit*

$$w^*\text{-} \lim_{\varepsilon \to 0+} \frac{1}{C_{\alpha,r}} \int_{\varepsilon}^{\infty} t^{-2\alpha}(C(t,A^{\odot}) - I)^r \overline{x}^* \frac{dt}{t};$$

in this case, this limit is $-(-A^{\odot})^{\alpha}\overline{x}^*$.

In [131], Fattorini has studied the relation between the domains of fractional powers of operators with a set of elements on which C_0-semigroups have fractional derivatives. Recall that a C_0-semigroup of operators has a continuous fractional derivative of order $\alpha \geq 0$ for $t \geq 0$ iff there exist $\beta > \omega(A)$ and a continuous function $f_\beta(\cdot)$, with the function $s^\alpha \|f_\beta(s)\|$ integrable in $s \geq 0$, and, moreover,

$$e^{-\beta t} \exp(tA)x = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \int_t^{\infty} (s-t)^{\alpha-1} f_\beta(s)\, ds, \quad t \geq 0. \tag{4.4}$$

Denote by $E_{\alpha,\beta}$ the set of elements $x \in E$ satisfying (4.4), and by F_α the set $\mathcal{D}((bI - A)^\alpha)$ for $b \geq \omega(A)$.

Proposition 4.3.3 ([131]). *Let $\alpha \geq 0$, and let $A \in \mathcal{G}(M,\omega)$. Then $E_{\alpha,\beta} = E_\alpha$, $\beta > \omega$.*

For C_0-groups of operators, the case of the previous proposition is complemented by one more relation $E^-_{\alpha,\beta} = F^-_\alpha$, $\beta > \omega$, where $F^-_\alpha = \mathcal{D}((bI + A)^\alpha)$ and $E^-_{\alpha,\beta}$ corresponds to $\exp(\cdot A)$.

Proposition 4.3.4 ([131]). *Let $\tau \geq 0, x \in E$, and let $0 < \alpha < \beta \leq 1$. Then $\int_0^\tau (\tau - s)^{2\beta-1} C(s,A)x\, ds \in \mathcal{D}((-A)^\alpha)$.*

As Fattorini has shown, it is not possible to set $\alpha = \beta = 1/2$ in the last proposition. This forces the appearance of Condition (F); see p. 98.

For a cosine operator function $C(\cdot, A)$, let us define the modulus of continuity as follows:

$$\omega_r(t^r, x) := \sup_{|s| \leq t} \|(C(s,A) - I)^r x\|_E, \quad x \in E,$$

where $r \in \mathbb{N}$. Also, we set $K(t,x;E,\mathcal{U}) = \inf_{g \in \mathcal{U}} \{\|x - g\|_E + t|g|_{\mathcal{U}}\}$.

Proposition 4.3.5 ([166]). *Let $r \in \mathbb{N}$, and let $0 < t \leq \delta < \infty$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 K(t^{2r}, x; E, \mathcal{D}(A^r)) \leq \omega_r(t^r, x) + \min(1, t^{2r})\|x\|_E \leq C_2 K(t^{2r}, x; E, \mathcal{D}(A^r)),$$

where K is the Peetre functional.

If $A \in \mathcal{C}(M,0)$, we can set $\delta = \infty$.

As for semigroups of operators, define $(E,\mathcal{D}(A))_{\theta,q}$ as the space with the norm

$$\|x\|_{(E,\mathcal{D}(A))_{\theta,q}} := \left(\int_0^\infty (t^{-\theta}K(t^r,x)^q\,dt\right)^{1/q}.$$

In the case $q=\infty$, the norm is given by

$$\|x\|_{(E,\mathcal{D}(A))_{\theta,\infty}} = \|x\|_E + \sup_{t\in\mathbb{R}_+}(t^{-2\alpha}\omega_1(t,x)).$$

Theorem 4.3.1 ([166]). *Let $0<\alpha<r$, $r\in\mathbb{N}$, $1\le q<\infty$ (or resp. $0\le\alpha\le r$, $q=\infty$). Then the intermediate spaces $(E,\mathcal{D}(A^r))_{\theta,q}$ with $\theta=\alpha/r$ and $0<\delta<\infty$ have the following equivalent norms:*

$$\text{(i)}\quad \left(\int_0^\delta t^{-\alpha/r}K(t,x;E;\mathcal{D}(A^r))^q\,\frac{dt}{t}\right)^{1/q};$$

$$\text{(ii)}\quad \|x\|_E + \left(\int_0^\delta (t^{-2\alpha}\omega_r(t^r,x))^q\,\frac{dt}{t}\right)^{1/q};$$

$$\text{(iii)}\quad \|x\|_E + \left(\int_0^\delta (t^{-2\alpha}\|(C(t,A)-I)^r x\|)^q\,\frac{dt}{t}\right)^{1/q}.$$

If $A\in C(M,0)$, then we can set $\delta=\infty$ in the previous theorem.

Corollary 4.3.1 ([166]). *Under the conditions of the previous theorem, we have*

(i) $C(t,A)(E,\mathcal{D}(A^r))_{\theta,q}\subseteq(E,\mathcal{D}(A^r))_{\theta,q}$, $t\in\mathbb{R}_+$, $0<\alpha<r$, $1\le q<\infty$ (*or* $0\le\alpha\le r$, $q=\infty$);

(ii) $S(t,A)(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},q}\subseteq(E,\mathcal{D}(A^r))_{\frac{\alpha+1/2}{r},q}$, $t\in\mathbb{R}_+$, $0<\alpha<r-1/2$, $1\le q<\infty$ (*or* $0\le\alpha\le r-1/2$, $q=\infty$).

Proposition 4.3.6 ([166]). *Let $A\in\mathcal{C}(M,0)$. Then*

(i) $(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},1}\subseteq\mathcal{D}((-A)^\alpha)\subseteq(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},\infty}$ *if* $r\in\mathbb{N}, 0<\alpha<r$;

(ii) $(E,\mathcal{D}(A^\beta))_{\theta,q)}\subseteq(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},q}$ *if* $r\in\mathbb{N}, 0<\alpha<\beta\le r, 1\le q\le\infty, \theta=\alpha/r$, *since the Favard class* $(E,\mathcal{D}((-A)^\alpha))_{1,\infty}$ *consists of elements with the norm*

$$\|x\| + \sup_{\varepsilon>0}\left\|\frac{1}{C_{\alpha,r}}\int_\varepsilon^\infty t^{-2\alpha}(C(t,A)-I)^r\frac{dt}{t}\right\|_E.$$

In particular, $\mathcal{D}((-A)^\beta)$ *is dense in* $(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},q}$ *for* $0<\alpha<\beta\le r, 1\le q<\infty$.

Let $(E,\mathcal{D}(A^r))^o_{\frac{\alpha}{r},q}$ denote the closure of $\mathcal{D}(A^r)$ in $(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},q}$.

Corollary 4.3.2 ([166]). *An element* $x\in(E,\mathcal{D}(A^r))^o_{\frac{\alpha}{r},q}$, $0\le\alpha\le r, r\in\mathbb{N}$, *iff* $\lim_{t\to0+}\|(C(t,A)-I)x\|_{(E,\mathcal{D}(A^r))_{\frac{\alpha}{r},q}}=0$.

Proposition 4.3.7. *Let $A \in \mathcal{C}(M,0)$. Then*

(i) *if $x \in (E, \mathcal{D}(A^r))_{\frac{\alpha}{r},q}$ with $0 < \alpha < r, 1 \le q < \infty$ (or $0 \le \alpha \le r, q = \infty$), $r \in \mathbb{N}$, then*

$$\|(C(t,A) - I)^r x\|_E = O(t^{2\alpha}), \quad t \to 0+. \tag{4.5}$$

Conversely, if (4.5) holds, then $x \in (E, \mathcal{D}(A^r))_{\frac{\alpha}{r},\infty}$;

(ii) *for $0 < \alpha < r, r \in \mathbb{N}$, an element $x \in (E, \mathcal{D}(A^r))^o_{\frac{\alpha}{r},\infty}$ iff*

$$\|(C(t,A) - I)^r x\|_E = o(t^{2\alpha}), \quad t \to 0+. \tag{4.6}$$

In [131], it was proved that for $A \in \mathcal{C}(M,0)$, $x \in \mathcal{D}(A^{1/2})$ implies $\sin(\sqrt{A}t)x \in \mathcal{D}(A^\gamma)$, $0 < \gamma < 1/2$, and

$$\|A^\gamma \sin(\sqrt{A}t)x\| \le ct^{1-2\gamma}\|\sqrt{A}x\|.$$

Proposition 4.3.8. *Let $A \in \mathcal{C}(M,0)$, and let E be reflexive. Then*

$$S(t,A)(E, \mathcal{D}(A))_{1/2,\infty} \subseteq \mathcal{D}(A), \quad t \in \mathbb{R}_+.$$

As in the previous section, denote by $E^-_{\alpha,\beta}, E^+_{\alpha,\beta}$ the subpaces related to $C(\cdot, A)$ as for C_0-groups early, i.e., the spaces of fractional derivatives.

Proposition 4.3.9 ([131]). *Let $\alpha \ge 0, \alpha \ne k + 1/2$, $k \in \mathbb{N}_0$. Then*

$$E^-_{2\alpha,\beta}, E^+_{2\alpha,\beta} \subseteq \mathcal{D}((bI - A)^\alpha), \quad \beta, b > \omega_c(A). \tag{4.7}$$

In the case $\alpha = k + 1/2$, $k \in \mathbb{N}_0$, inclusion (4.7) can violate.

Theorem 4.3.2 ([131]). *Let $E = L^p(X, \Sigma, \mu)$ with $1 < p < \infty, A \in \mathcal{C}(M, \omega)$, and let $u^0 \in \mathcal{D}((bI - A)^\alpha)$, $u^1 \in \mathcal{D}((bI - A)^\gamma)$, $\gamma = \max\{\alpha - 1/2, 0\}$. Then for a solution of problem (3.1), we have*

(i) *if $0 \le \alpha \le 1$, then $\|u(t) - u(0)\| = O(t^{2\alpha})$, $t \to 0+$;*

(ii) *if $1/2 \le \alpha \le 1$, then $u(\cdot)$ is continuously differentiable and $\|u'(t) - u'(0)\| = O(t^{2\alpha-1})$, $t \to 0+$.*

To obtain the assertion of the theorem in the general Banach space E, we need an additional smoothness, i.e., $u^0 \in \mathcal{D}((bI - A)^{\alpha+\delta})$, $u^1 \in \mathcal{D}((bI - A)^\gamma)$, $\gamma = \max\{\alpha + \delta - 1/2, 0\}$ for a certain $\delta > 0$.

Definition 4.3.1. For an operator $A \in \mathcal{C}(E)$, we set

$$\widetilde{\mathcal{D}(A)}^E := \{x \in E : \exists \{x_n\} \subseteq \mathcal{D}(A) \text{ such that } \|x_n\|_{\mathcal{D}(A)} \le M \text{ and } \lim_{n\to\infty} \|x_n - x\|_E = 0\}.$$

In the operator semigroup theory, $\widetilde{\mathcal{D}(A)}^E$ is the Favard class (saturation class).

Proposition 4.3.10. *Let $A \in \mathcal{C}(E)$. Then for $t \to 0+$,*

$$K(t, x; E, \mathcal{D}(A)) = \begin{cases} O(t) & \text{for } x \in \widetilde{\mathcal{D}(A)}^E, \\ o(t) & \text{for } x \in \mathcal{N}(A). \end{cases}$$

Moreover, if E is reflexive, then $\mathcal{D}(A) = \widetilde{\mathcal{D}(A)}^E$.

We set $C_t^\beta[C(\cdot, A)] = \dfrac{\beta}{t^\beta} \int\limits_0^t (t-s)^{\beta-1} C(s, A)\, ds$.

Proposition 4.3.11. *Let $A \in \mathcal{C}(M, \omega)$. Then the following conditions are equivalent for $0 < \alpha \leq 2$:*

$$\text{(i)} \quad \|C_t^\alpha[C(\cdot, A)]x - x\|_E = O(t^\alpha), \quad t \to 0;$$

$$\text{(ii)} \quad K(t^\alpha, x, E; \mathcal{D}(A)) = O(t^\alpha), \quad t \to 0.$$

Moreover, the condition $x \in \mathcal{N}(A)$ is equivalent to $\|C_t^\alpha[C(\cdot, A)]x - x\|_E = o(t^2)$, $t \to 0$.

With the notation $u'' = (A - c^2 I)u(t) = B^2 u(t)$, $u(0) = x, u'(0) = y$ for $c = 0$, we have the following assertion.

Theorem 4.3.3 ([122]). *Let $x \in E$ be such that $\|C(t, A)x - x\| = o(t^2)$ as $t \to 0$. Then $x \in \mathcal{D}(B^2)$ and $B^2 x = 0$. The saturation $\|C(t, A)x - x\| = O(t^2)$, $t \to 0+$, holds iff $x \in \widetilde{\mathcal{D}(B^2)}^E$. If E is reflexive, then $\widetilde{\mathcal{D}(B^2)}^E = \mathcal{D}(B^2)$.*

Denote

$$V(t)x = \frac{1}{t} \int_0^t C(s, A) x ds.$$

Theorem 4.3.4 ([122]). *Let $x \in E$ be such that $\|V(t)x - x\| = o(t^2)$, $t \to 0+$. Then $x \in \mathcal{D}(B^2)$ and $B^2 x = 0$. The saturation $\|V(t)x - x\| = O(t^2)$, $t \to 0+$, holds iff $x \in \widetilde{\mathcal{D}(B^2)}^E$.*

If $c \neq 0$, then we have the operator $B^2 + c^2 I$ in the saturation theorem. For example, the following theorem holds.

Theorem 4.3.5 ([122]). *Let x and y be such that $\|t^{-1}(u(t) - x) - y\| = o(t^2)$, $t \to 0$. Then $x \in \mathcal{D}(B^2)$, $y \in \mathcal{D}(B^2)$, $(B^2 + c^2 I)x = (B^2 + c^2 I)y = 0$ and $u(t) = x + ty$. Moreover, $\|t^{-1}(u(t) - x) - y\| = O(t^2)$, $t \to 0$ iff $x \in \mathcal{D}(B^2)$, $(B^2 + c^2 I)x = 0$, and $y \in \widetilde{\mathcal{D}(B^2)}^E$.*

Proposition 4.3.12 ([252]). *For $A \in \mathcal{C}(M, 0)$, we have the Landau inequalities*

$$\|A^2 x\|^4 \leq \frac{1024}{315} \|x\|^3 \|A^4 x\|, \ \|A^2 x\|^4 \leq \frac{400}{49} \|x\|^2 \|A^4 x\|^2, \ \|A^3 x\|^4 \leq \frac{2880}{343} \|x\| \|A^4 x\|^3.$$

SPECTRAL PROPERTIES OF C_0-COSINE OPERATOR FUNCTIONS

In the same way as for a C_0-semigroup, necessary and sufficient conditions for A to generate a C_0-cosine operator function are formulated in terms of conditions on the location of the spectrum and estimates for the resolvent; see [17]. For a narrow class of C_0-cosine operator functions on a Hilbert space, these conditions are essentially based on the location of the spectrum; see [210].

5.1. Location of the Spectrum

Proposition 5.1.1 ([221]). *Let a C_0-cosine operator function $C(\cdot, A)$ be given. Then*

$$\text{(i)} \quad \cosh(t\sqrt{\sigma(A)}) \subseteq \sigma(C(t,A)), \quad t \in \mathbb{R};$$

$$\text{(ii)} \quad \cosh(t\sqrt{P\sigma(A)}) = P\sigma(C(t,A)), \quad t \in \mathbb{R};$$

$$\text{(iii)} \quad \cosh(t\sqrt{R\sigma(A)}) \subseteq R\sigma(C(t,A)), \quad t \in \mathbb{R}.$$

Proposition 5.1.2 ([5, 222]). *If $\mu \in R\sigma(C(t,A))$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ is the set of roots of the equation $\mu = \cosh(\lambda_n t)$, then $\lambda_{n_0}^2 \in R\sigma(A)$ for a certain $n_0 \in \mathbb{N}$, and $\lambda_n^2 \notin P\sigma(A)$ does not hold for any $n \in \mathbb{N}$ $\mu \in P\sigma(C(t,A)^*)$.*

Proposition 5.1.3 ([5, 221]). *If $\mu \in C\sigma(C(t,A))$ and λ_n are from Proposition* 5.1.2, *then $\lambda_n^2 \in C\sigma(A) \cup \rho(A)$. The case where $\lambda_n^2 \in \rho(A)$ for all $n \in \mathbb{N}$ is possible.*

Proposition 5.1.4 ([139, 181]). *If $E = H$ is Hilbert and $A \in \mathcal{C}(M,0)$ or $C(\cdot, A)$ is a family of normal operators, then*

$$\sigma(C(t,A)) = \cosh(t\sqrt{\sigma(A)}), \quad t \in \mathbb{R}.$$

Proposition 5.1.5 ([105]). *Let a C_0-cosine operator function $C(\cdot, A)$ satisfy Condition* (F), *and let $E = H$ be Hilbert. Then $\mu \in \rho(C(t,A))$ iff $\{z^2 : \cosh(zt) = \mu\} \subseteq \rho(A)$ and $\sup\{\|z\mathcal{R}(z^2; A)\| : \cosh(zt) = \mu\} < \infty$.*

Proposition 5.1.6 ([206]). *Let $A \in \mathcal{C}(M,0)$. Then*

(i) $\sigma(A) \subset \mathbb{R}_-$;

(ii) *if $E \neq \{0\}$, then $\sigma(A) \neq \varnothing$;*

(iii) *the spectrum $\sigma(A)$ is bounded iff $A \in B(E)$.*

A Banach space E is said to be *hereditarily indecomposable* (in brief, an H.I. space) if whenever X_1 and X_2 are closed infinite-dimensional subspaces of E and $\delta > 0$, then there exist unit vectors $x_1 \in X_1, x_2 \in X_2$

such that $\|x_1 - x_2\| < \delta$. In other words, this property can be reformulated as follows [147]: for any two infinite-dimensional subspaces $X_1, X_2 \subset E$ such that $X_1 \cap X_2 = \{0\}$, the subspace $X_1 + X_2$ is not closed.

Proposition 5.1.7 ([248]). *Let E be an H.I. space, and let $A \in C(M, \omega)$. Then $\sigma(A)$ is either a finite set (possibly empty) in $\mathbb{C}$ or consists of a sequence $\{\mu_n\}_{n=1}^{\infty}$ that either converges to a certain point of $\mathbb{C}$ or satisfies $\lim\limits_{n\to\infty} \operatorname{Re} \mu_n = -\infty$.*

Proposition 5.1.8 ([248]). *Let E be an H.I. space, and let $\mathcal{C}(\cdot, A)$ be a non-quasi-analytic C_0-cosine operator family. Then $\sigma(A) \cap \mathbb{C} \neq \varnothing$.*

Proposition 5.1.9. *Let $A \in \mathcal{GR}(M, \omega)$. Then*

(i) *the spectrum of A lies in the strip $-\omega < \operatorname{Re} z < \omega$ (see Fig. 2);*

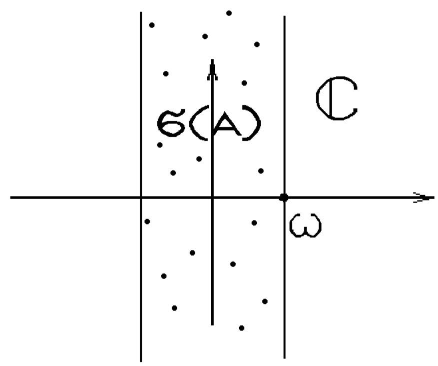

Fig. 2

(ii) *the operator A^2 generates a C_0-cosine operator function by the formula*

$$C(t, A^2) = \frac{1}{2}\Big(\exp(tA) + \exp(-tA)\Big), \quad t \in \mathbb{R}.$$

Proposition 5.1.10 ([15]). *For $A \in \mathcal{C}(M, \omega)$ and the corresponding matrix differential operator $\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ arising in reducing the Cauchy problem (3.1) to system (3.5), the following relation holds:*

$$\{\lambda^2 : \lambda \in \sigma(\mathcal{A})\} = \sigma(A).$$

Proposition 5.1.11 ([221]). *Let $A \in \mathcal{C}(M, \omega)$. Then the spectrum $\sigma(A)$ lies on a certain parabola whose branches are directed to the left; see Fig. 3.*

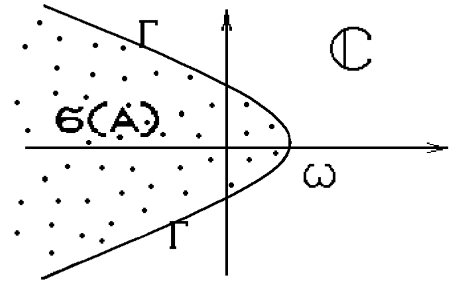

Fig. 3

Proposition 5.1.12 ([5]). *There exist a C_0-cosine operator function $C(\cdot, A)$ and a Banach space E such that the sets $r_1 := \{t : 0 \in \rho(C(t, A))\}$, $r_2 := \{t : 0 \in P\sigma(C(t, A))\}$, and $r_3 := \{t : 0 \in C\sigma(C(t, A))\}$, are dense in $\mathbb{R}$, and, moreover, $\mathbb{R} = r_1 \cup r_2 \cup r_3$.*

Proposition 5.1.13 ([86]). *For the C_0-cosine operator function $C(t, A) = \sum\limits_{k=0}^{\infty} \dfrac{t^{2k}}{(2k)!} A^k$, $A \in \mathcal{B}$, given on a Banach algebra $\mathcal{B}$ with unity, we have*

(i) $0 \leq \omega_c(A) < \infty$;

(ii)
$$\lambda R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} C(t, A)\, dt \quad \textit{for} \quad \operatorname{Re} \lambda > \omega_c(A);$$

(iii)
$$R(\lambda^2, A) = \int_0^\infty e^{-\lambda t} S(t, A)\, dt \quad \textit{for} \quad \operatorname{Re} \lambda > \omega_c(A);$$

(iv)
$$C(t, A) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} \lambda (\lambda^2 I - A)^{-1}\, d\lambda, \quad t \in \mathbb{R};$$

(v)
$$S(t, A) = \frac{1}{2\pi i} \int_\gamma e^{\lambda t} (\lambda^2 I - A)^{-1}\, d\lambda, \quad t \in \mathbb{R},$$

where γ is a certain contour enclosing the spectrum of the operator $A \in \mathcal{B}$.

Proposition 5.1.14 ([86]). *Under the conditions of Proposition* 5.1.13, *we have*

$$\omega_c(A)^2 = \sup_{\lambda \in \sigma(A)} (|\lambda| + \operatorname{Re} \lambda)/2.$$

Theorem 5.1.1 ([106]). *Let $A \in \mathcal{C}(M, \omega)$. The following conditions are equivalent:*

(i) $1 \in \rho(C(2\pi, A))$;

(ii) $-\mathbb{N}_0^2 \subseteq \rho(A)$ *and the sequences*

$$R_N = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} (-k^2 I - A)^{-1}$$

and

$$S_N = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} A(-k^2 I - A)^{-1}$$

are bounded in $B(E)$;

(iii) $-\mathbb{N}_0^2 \subseteq \rho(A)$ *and there exist the limits*

$$Rx := s\text{-}\lim_{N\to\infty} R_N x \quad \text{and} \quad Sx := s\text{-}\lim_{N\to\infty} S_N x \tag{5.1}$$

for all $x \in E$.

Theorem 5.1.2 ([106]). *Let* $A \in \mathcal{C}(M,\omega)$. *In a Hilbert space* $E = H$, *the following condition are equivalent*:

(i) $1 \in \rho(C(2\pi; A))$;

(ii) $-\mathbb{N}_0^2 \subseteq \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|k(-k^2 I - A)^{-1}\| < \infty$.

Chapter 6

UNIFORMLY BOUNDED C_0-COSINE OPERATOR FUNCTIONS

In this chapter, we collect assertions that, in one way or another, are related to the boundedness of cosine operator functions, although we consider C_0-cosine operator functions of polynomial and sometimes exponential growth. The matter is that the asymptotic behavior of resolving families for second order equations differs from that of operator semigroups, and the representation $C(t,A) = \frac{1}{2}(\exp(t\sqrt{A}) + \exp(-t\sqrt{A}))$ does not always hold.

6.1. Behavior of C_0-Cosine Operator Functions at Infinity

As was already noted, the boundedness of a C_0-cosine operator function is a property that is not obtained by a shift of the generator $A_b = A + bI$.

Proposition 6.1.1 ([144]). *There exist operators* $A \in \mathcal{C}(M,\omega)$, *such that for any number* $b \in \mathbb{R}$, *the operator* $A + bI$ *does not generate a bounded* C_0*-cosine operator function.*

With the cosine equation (i) (see p. 83) one associates the hyperbolic cosine $\cosh(t)$ of exponential growth, as well as the bounded ordinary function $\cos(t)$. In the general case, for a C_0-cosine operator function, a polynomial growth in t is also possible. So, for example, we have the following.

Example 6.1.1 ([207]). Let $E = \mathbb{R}^2$. Then $C(t) = \begin{pmatrix} 1 & 0 \\ \frac{t^2}{2} & 1 \end{pmatrix}$, $t \in \mathbb{R}$, is a C_0-cosine operator function on E (i.e., (i) on p. 83 holds). If the Euclidean norm is given on E, then

$$\|C(t)\| = \sqrt{1 + \frac{t^2}{2} + \frac{t^4}{4}}, \quad t \in \mathbb{R}.$$

Under the conditions of the previous example, for any $\omega > 0$, there exists $M_\omega \geq 1$ such that $\|C(t)\| \leq M_\omega \cosh(\omega t)$, $t \in \mathbb{R}$, but $C(\cdot)$ is not bounded on $\mathbb{R}$.

Note by the way that Example 6.1.1 describes exactly the case where there arises the problem on the representation of $C(\cdot)$ as the half-sum of two exponential functions [109].

Example 6.1.2 ([119]). Let A be the operator on the space $C^1([a, b])$ defined by the formula

$$(Af)(s) := sf(s), \quad s \in [a, b].$$

Then A is a generator of a C_0-family of a cosine operator function $(C(t, A)f)(s) = h(s, t)f(s)$, $s \in [a, b]$, $t \in \mathbb{R}$, where $h(s, t) = \cos(t\sqrt{-s})$, $s \in [a, b]$, $t \in \mathbb{R}$. The norm of this C_0-cosine operator function is equal to

$$\|C(t, A)\| = \max \left\{ \sup_{s \in [a,b]} |h(s, t)|, \sup_{s \in [a,b]} \left| \frac{d}{ds} h(s, t) \right| \right\}$$

and admits the estimate

$$\|C(t_n, A)\| \geq c(1 + |t_n|)$$

with certain $\{t_n\}$, $\lim_{n \to \infty} t_n = \infty$, and a constant $c > 0$.

Therefore, the spectrum of the operator A coincides with the closed interval $[a, b]$, and for any $a, b < 0$, the norm $\|C(t, A)\|$ is not bounded on $\mathbb{R}$.

Example 6.1.3 ([119]). Let A be the operator on the space $L^1(\mathbb{R})$ defined by the formula

$$(Af)(s) := \left(\frac{d}{ds} \right)^2 f(s), \quad s \in \mathbb{R}.$$

Then A is a generator of the C_0-cosine operator function $(C(t, A)f)(s) = \frac{1}{2}\Big(f(t + s) + f(t - s)\Big)$, $s, t \in \mathbb{R}$. Moreover, the C_0-semigroup generated by A admits the estimate

$$\|\exp(zA)\| \leq \left(\frac{|z|}{\mathrm{Re}(z)} \right)^{\frac{1}{2}} \quad \text{for any } z \text{ with } \mathrm{Re}\, z \geq 0.$$

Example 6.1.4 ([119]). Let A be the operator on the space $L^p(\mathbb{R}^d)$ defined by the formula

$$(Af)(s) := \Delta f(s), \quad s \in \mathbb{R}^d, \quad d \in \mathbb{N}.$$

Then A is a generator of a C_0-cosine operator function $C(\cdot, A)$ only in the case $p = 2$ in general. Moreover, the C_0-semigroup generated by the operator A admits the estimate

$$\|\exp(zA)\| \leq \left(\frac{|z|}{\operatorname{Re}(z)}\right)^{d|\frac{1}{p}-\frac{1}{2}|} \quad \text{for any } z \quad \text{with} \quad \operatorname{Re} z \geq 0.$$

Theorem 6.1.1 ([119]). *The following implications of conditions hold*: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

(i) *A function $C(\cdot, A)x$ is of exponential type less than or equal to ω for all $x \in E$*;

(ii) $\{z^2 : \operatorname{Re} z > \omega\} \subseteq \rho(A)$, *and for any $\gamma > \omega$, there exists a constant $M = M(\gamma)$ such that*

$$\|z(z^2 I - A)^{-1}\| \leq M \quad \textit{whenever } \operatorname{Re} z > \gamma;$$

(iii) *the function $C(\cdot, A)x$ is of exponential type not exceeding ω for all $x \in D(A)$.*

Theorem 6.1.2 ([119]). *The following implications of conditions hold*: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

(i) *a C_0-cosine operator function $C(\cdot, A)$ is bounded*;

(ii) *there exists a constant M_1 such that*

$$\|\exp(zA)\| \leq M_1 \left(\frac{|z|}{\operatorname{Re}(z)}\right)^{\frac{1}{2}} \quad \textit{for any } z \textit{ with } \operatorname{Re} z \geq 0;$$

(iii) *there exists a constant M_2 such that*

$$\|C(t, A)x\| \leq M_2(\|x\| + t^2\|Ax\|), \quad t \in \mathbb{R}, \quad x \in D(A).$$

An attempt to give necessary and sufficient conditions for the uniform boundedness of C_0-cosine operator functions was undertaken in [108], but, as K. Bojadzhiev showed, the proof contains inaccuracies.

6.2. Uniformly Bounded C_0-Cosine Operator Functions

For all $x \in E$ and $a \in \overline{\mathbb{R}}_+$, let us define the operator

$$F_a x := \frac{2}{\pi} \int_0^\infty \left(\frac{\sin(at)}{t}\right)^2 C(2t, A)x dt,$$

which, obviously, is bounded for $A \in \mathcal{C}(M, 0)$.

Proposition 6.2.1 ([245]). *Let $0 \leq a \leq b$. Then*

$$F_a F_b x = F_b F_a x = 2\int_0^a F_u x du + (b - a)F_a x, \quad x \in E.$$

Proposition 6.2.2 ([245]). *For certain $0 \leq a \leq b$, let the following relation hold*:

$$2\int_a^b F_t x dt = (b - a)(F_a x + F_b x)$$

for any $x \in E$. Then the open interval $(-b^2, -a^2) \subseteq \rho(A)$.

Proposition 6.2.3 ([245]). *For the operator F_a, the following relations hold for $a \in \mathbb{R}_+$:*

$$F_a^k = (k-1)k \int_0^a (a-t)^{k-2} F_t dt, \quad k = 2, 3, ..., \quad \exp(itF_a) = I + itF_a - t^2 \int_0^a e^{it(a-s)} F_s\, ds.$$

Let a C_0-cosine operator function $C(\cdot, A)$ be such that the operator

$$E_a x := s\text{-}\lim_{\alpha \to 0+} E_{a,\alpha} x := s\text{-}\lim_{\alpha \to 0+} \int\limits_{\alpha+i0}^{\alpha+ia} \Big(\lambda(\lambda^2 I - A)^{-1} + \overline{\lambda} R(\overline{\lambda}^2 I - A)^{-1} \Big) x d\lambda \tag{6.1}$$

(where $\lambda = \alpha + i\tau$) is linear and continuous for all $a \in \overline{\mathbb{R}}_+$.

Proposition 6.2.4 ([245]). *There are examples of uniformly bounded C_0-cosine operator functions for which the family $\{E_a\}$ from* (6.1) *is not defined.*

Proposition 6.2.5 ([245]). *For a C_0-cosine operator function $C(\cdot, A)$, let family* (6.1) *be defined, and let $E_a = E_b$ for certain $0 < a < b$. Then*

$$(-b^2, -a^2) \cap \Big(R\sigma(A) \cup P\sigma(A) \Big) = \varnothing.$$

Proposition 6.2.6 ([245]). *Let the function $E_{a,\alpha} x$ be bounded for $a \in [0, \overline{a}]$ and $\alpha \in [0, \overline{\alpha}]$ with any $\overline{a}, \overline{\alpha} \geq 0, \overline{\alpha} \neq 0$. Then for all $x \in E$ and $a \in [0, \overline{a}]$, there exists $E_a x$, the operator E_a is bounded, and for all $0 \leq a \leq b$, the relation $E_a E_b = E_b E_a = \pi i E_a$ holds; moreover, for almost all $a \in \overline{\mathbb{R}}_+$, we have the relation $E_a x = \frac{2}{\pi} \int\limits_0^\infty \frac{\sin(at)}{t} C(t, A) x dt$ in the case where the integral converges.*

Further, for $0 \leq a \leq b$ denote $\Delta := (-b^2, -a^2)$ and $E_\Delta := E_b - E_a$.

Proposition 6.2.7 ([245]). *Under conditions of Proposition* 6.2.6*, for any two intervals Δ_1 and Δ_2, we have*

$$E_{\Delta_1} E_{\Delta_2} = E_{\Delta_2} E_{\Delta_1} = E_{\Delta_1 \cap \Delta_2}.$$

Proposition 6.2.8 ([245]). *Let $\|E_{\psi,\varphi} x\| \leq \mathrm{const}$ for any $\psi \in \overline{\mathbb{R}}_+$ and $\varphi \in \mathbb{R}_+$. Moreover, let $E_a = E_b$ for certain $0 \leq a \leq b$. Then $(-b^2, -a^2) \subseteq \rho(A)$.*

Proposition 6.2.9 ([245]). *Under the conditions of Proposition* 6.2.6*, let the function $E_a x$ be continuous at the point $a_0 \in \overline{\mathbb{R}}_+$ for any $x \in E$. Then $-a_0^2 \notin R\sigma(A)$.*

Proposition 6.2.10 ([245]). *Let the space E be reflexive and strongly convex with the Gateaux-differentiable norm, $A \in \mathcal{C}(M, 0)$, and let the operator $C(t, A)$ have a real spectrum for any $t \in \mathbb{R}$. Then $R\sigma(A) = \varnothing$.*

Proposition 6.2.11 ([14]). *Let $A \in \mathcal{C}(1,0)$, and let $\mathcal{A}$ be the operator from Theorem* 3.2.1. *Then for $t \in [\ln 2, \infty)$,*

$$\|C'(t,A)\|_{B(E^1,E)} \le t \cdot \ln 2, \ \|S(t,A)\|_{B(E,E^1)} \le t+1,$$

and the resolvent $(\lambda I - \mathcal{A})^{-1}$ satisfies the estimate

$$\|R(\lambda, \mathcal{A})\| \le (\operatorname{Re}\lambda - \ln 2)^{-1} \quad \textit{for} \qquad \operatorname{Re}\lambda > \ln 2.$$

Proposition 6.2.12 ([6,7]). *Let $0 \notin \sigma(A)$, and let $A \in \mathcal{C}(M,0)$. Then*

$$\sup_{t\in\mathbb{R}} \|S(t,A)\| \le \frac{\pi}{2} \operatorname{dist}(0, \sqrt{\sigma(A)}) \sup_{t\in\mathbb{R}} \|C(t,A)\|.$$

Proposition 6.2.13 ([6,7]). *For $A \in \mathcal{C}(M,0)$, the set $\{x \in E : \sup\limits_{t\in\overline{\mathbb{R}}_+} \|S(t,A)x\| < \infty\}$ is dense in E iff one of the following conditions holds*:

(i) *s-$\lim\limits_{n\to\infty} \varepsilon_n R(\varepsilon_n, A)x = 0$ for any $x \in E$ and a certain sequence $\varepsilon_n \in \mathbb{R}_+$ such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$;*

(ii) *the set $\mathcal{R}(A)$ is dense in E;*

(iii) $\mathcal{N}(A^*) = \{0\}$.

Proposition 6.2.14 ([108]). *An operator $A \in \mathcal{C}(M,0)$ satisfies Condition* (F) *iff the following condition holds for each closed interval $[a,b]$*:

$$\sup\left\{ \| \exp(tG|_{D(G,\mu)})\| \ : \mu \in \mathbb{N}, t \in [a,b] \right\} < \infty.$$

Proposition 6.2.15 ([177]). *Let $A \in \mathcal{C}(M,0)$. Then the operator iA generates an α times integrated group for $\alpha > \frac{1}{2}$.*

Proposition 6.2.16 ([139]). *Let $A \in \mathcal{C}(M,0)$, and let $E = H$. Then there exist a self-adjoint operator Q and a constant $M > 0$ such that $(\sqrt{3}(2M+1))^{-1} I \le Q \le MI$ and the operator $QC(t,A)Q^{-1}$ is self-adjoint for each $t \in \mathbb{R}$. Moreover, $C(t,A) = Q^{-1}\cos(tL)Q$ and $L^* = L \le 0$, where $L := QAQ^{-1}$.*

Proposition 6.2.17 ([146]). *For $A \in \mathcal{C}(M,0)$ and $x \in D(A)$, the following inequality holds*:

$$\sup_{t\in\overline{\mathbb{R}}_+} \|S(t,A)Ax\|^2 \le 4 \sup_{t\in\overline{\mathbb{R}}_+} \|C(t,A)Ax\| \cdot \sup_{t\in\overline{\mathbb{R}}_+} \|C(t,A)x\|.$$

Proposition 6.2.18 ([206]). *Let $A \in \mathcal{C}(1,0)$, and let*

$$C_m(t) := \sum_{j=0}^{m} C_{2m}^{2j} \left(\frac{t}{2m}\right)^{2j} A^j \left(\frac{2m}{t}\right)^{4m} \left(\left(\frac{2m}{t}\right)^2 I - A\right)^{-2m},$$

where C_{2m}^{2j} are binomial coefficients. Then $\lim\limits_{m\to\infty} C_m(t)x = C(t,A)x$ for all $x \in E$ and $\|(C_m(t) - C(t,A))x\| \le t^2\|Ax\|/\sqrt{m}$ for all $x \in D(A)$, $m \ge 2$.

Proposition 6.2.19 ([221]). *The functions $C(\cdot, A)x$ and $S(\cdot, A)x$ are uniformly bounded for any $x \in E$ iff there exists a constant $M \geq 1$ such that for* $\operatorname{Re} z \geq \omega$,

$$\left\| \frac{d^k}{dz^k}(z^2 I - A)^{-1} \right\|, \quad \left\| z \frac{d^k}{dz^k} R(z^2 I - A)^{-1} + k \frac{d^{k-1}}{dz^{k-1}}(z^2 I - A)^{-1} \right\| \leq \frac{Mk!}{|z|^{k+1}}, \quad k \in \mathbb{N}.$$

Proposition 6.2.20 ([177]). *Let an operator A generate a C_0-cosine operator function such that for the corresponding C_0-sine operator function $S(\cdot, A)$ the estimate $\|S(t, A)\| \leq Mt$, $t \in \overline{\mathbb{R}_+}$ holds. Then the operator iA generates an α times integrated semigroup for $\alpha > \frac{3}{2}$.*

6.3. Asymptotics of the Functions $F(\cdot)$ and $G(\cdot)$

In this section, we study the asymptotic behavior of the C_0-families $F(t)$ and $G(t)$ as $t \to \infty$. Let us consider the case under the assumption that there exist a real λ_0 and a nonzero bounded operator $P \in B(E)$ such that

$$\lim_{t \to \infty} 2e^{-\lambda_0 t}\, C(t, A)x = Px \quad \text{for all} \quad x \in E. \tag{6.2}$$

Clearly, in this case, there exists a constant $M_1 \geq 1$ such that

$$\|C(t, A)\| \leq M_1 e^{\lambda_0 t} \quad \text{for} \quad t \in \overline{\mathbb{R}}_+. \tag{6.3}$$

By the identity

$$2e^{-\lambda_0 2t}(C(2t, A) + I) = 2e^{-\lambda_0 t} C(t, A) 2e^{-\lambda_0 t} C(t, A), \tag{6.4}$$

in case (6.2), the number λ_0 cannot be negative, since then $e^{-\lambda_0 2t} \to \infty$ as $t \to \infty$, and there is no convergence.

In the case $\lambda_0 = 0$, we have from (6.4) that $P + 2I = P^2$. On the other hand, setting $t \to \infty$ and $s \to \infty$ in the cosine equation (see (i) on p. 83), we have $2P = P^2$. Therefore, $C(t, A) \to P/2 = I$ as $t \to \infty$. Setting $t \to \infty$ in relation (i) on p. 83, we obtain $C(s, A) = I$ for all $s \in \mathbb{R}$.

In connection with these simple arguments, we note that in [81], the assumption on the convergence of $C(t, A)$ to P as $t \to \infty$, which has no meaning, was made.

We note by the way that in the case $C(t, A) \to P$ as $t \to \infty$, it follows from (iv) of Proposition 2.4.1 that $F(t)x = 2^{-1} t^2 \lambda^3 \hat{F}(\lambda)x$ (for any $\lambda > \omega$), which never converges as $t \to \infty$ if $x \notin \mathcal{N}(\hat{F}(\lambda))$. The same situation takes place for $G(\cdot)$.

Therefore, the case $\lambda_0 = 0$ is not interesting, and in what follows, we will assume that $\lambda_0 > 0$. It is clear from (6.4) that the operator P is a projection.

It is known that a generator of a C_0-cosine operator function $C(\cdot, A)$ also generates a C_0-semigroup $\exp(\cdot A)$ defined by formula (2.12).

As will be shown in the following theorem, the convergence of $2e^{-\lambda_0 t}\, C(t, A)$ to P as $t \to \infty$ implies the convergence of $e^{-\lambda_0^2 t} \exp(tA)$ to P in the same topology.

Theorem 6.3.1. *Let condition* (6.2) *hold with* $\lambda_0 \in \mathbb{R}_+$. *Then P is a projection with the range* $\mathcal{R}(P) = \mathcal{N}(\lambda_0^2 I - A)$ *and the kernel* $\mathcal{N}(P) = \overline{\mathcal{R}(\lambda_0^2 I - A)}$. *If, moreover, P is of finite rank and* $\|2e^{-\lambda_0 t} C(t, A) - P\| \to 0$ *as* $t \to \infty$, *then* $\lambda_0^2 > E\omega(A)$, $B\sigma(A) = \{\lambda_0^2\}$, *and λ_0^2 is a simple pole of the resolvent* $(\lambda I - A)^{-1}$.

Proof. We prove that the C_0-semigroup $e^{-\lambda_0^2 t} \exp(tA)$ strongly converges to P as $t \to \infty$. Then it follows from the ergodic theorem (see [76]) that P is a projection with $\mathcal{R}(P) = \mathcal{N}(\lambda_0^2 I - A)$ and $\mathcal{N}(P) = \overline{\mathcal{R}(\lambda_0^2 I - A)}$.

We have

$$e^{-\lambda_0^2 t} \exp(tA)x = \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} (2e^{-\lambda_0 s} C(s, A) - P)x \, ds + \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} \, ds Px.$$

The first term $Q_1(t)$ on the right-hand side converges to zero, and the second term $Q_2(t)$ converges to Px as $t \to \infty$. Indeed,

$$\frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} ds = \frac{1}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{(s-2\lambda_0 t)^2}{4t}} ds = \frac{1}{\sqrt{\pi}} \int_{-\lambda_0\sqrt{t}}^\infty e^{-u^2} du$$

converges to $\frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} e^{-u^2} du = 1$ as $t \to \infty$. Therefore, $Q_2(t)$ converges to Px as $t \to \infty$.

Let $\epsilon > 0$ be sufficiently small, and let $\tau \in \mathbb{R}_+$ be so large that $\|2e^{-\lambda_0 s} C(s, A)x - Px\| \leq \epsilon$ for all $s \geq \tau$. Then the quantity

$$\|Q_1(t)\| \leq \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\tau e^{-\frac{s^2}{4t}} e^{\lambda_0 s} \|(2e^{-\lambda_0 s} C(s, A) - P)x\| ds + \epsilon \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} e^{\lambda_0 s} ds \|x\|,$$

and hence is bounded by the constant 2ϵ as $t \to \infty$. That is, $Q_1(t) \to 0$ as $t \to \infty$.

If $\|2e^{-\lambda_0 t} C(t, A) - P\| \to 0$ as $t \to \infty$, then in a similar way, we prove that $\|e^{-\lambda_0^2 t} \exp(tA) - P\| \to 0$ as $t \to \infty$. When P is of finite rank, the semigroup $\exp(\cdot A)$ attains the limit with the growth exponent λ_0^2. It follows from the theorem in [291] that $\lambda_0^2 > E\omega(A)$, $B\sigma(A) = \{\lambda_0^2\}$, and λ_0^2 is a simple pole of the resolvent $(\lambda I - A)^{-1}$.

We need the following assertion.

Lemma 6.3.1 (see [17, Lemma 7.3.1]). *If a strongly continuous function* $f(\cdot) : \overline{\mathbb{R}}_+ \to E$ *is such that* $\lim\limits_{t\to\infty} f(t) = \varphi$, $\varphi \in E$, *then for any λ with* $\operatorname{Re}\lambda > 0$, *we have*

$$e^{-\lambda t} \int_0^t e^{\lambda s} f(s) \, ds \to \varphi/\lambda \quad \text{for} \quad t \to \infty. \tag{6.5}$$

Proposition 6.3.1. *Let a C_0-cosine operator function $C(\cdot, A)$ satisfy (6.2) with $\lambda_0 > 0$. Then $2e^{-\lambda_0 t}S(t, A) \to P/\lambda_0$ and $2e^{-\lambda_0 t}\int_0^t S(s, A)ds \to P/\lambda_0^2$ strongly as $t \to \infty$.*

Proof. We have the following relations:

$$2e^{-\lambda_0 t}S(t, A) = e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s}2e^{-\lambda_0 s}C(s, A)\, ds$$

and

$$2e^{-\lambda_0 t}\int_0^t S(s, A)ds = e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s}e^{-\lambda_0 s}\int_0^s e^{\lambda_0 \eta}2e^{-\lambda_0 \eta}C(\eta, A)\; d\eta\; ds.$$

Now the assertion follows from Lemma 6.3.1.

Theorem 6.3.2 ([239]). *Let a C_0-cosine operator function $C(\cdot, A)$ satisfy condition (6.2) with $\lambda_0 \in \mathbb{R}_+$. Then for each $\lambda > \lambda_0$ and each $x \in E$, we have*

$$s\text{-}\lim_{t\to\infty} 2e^{-\lambda_0 t}F(t)x = \lambda(\lambda^2/\lambda_0^2 - 1)P\hat{F}(\lambda)x,$$

which is equal to $\frac{1}{\lambda_0^2}(A_s - A)x$ if $x \in D(A)$ and $\hat{F}(\lambda)x \in \mathcal{N}(\lambda_0^2 I - A)^{-1}$, simultaneously, and is equal to zero if $\hat{F}(\lambda)x \in \overline{\mathcal{R}(\lambda_0^2 I - A)^{-1}}$.

Proof. For $\lambda_0 \in \mathbb{R}_+$, let us write the relation

$$2e^{-\lambda_0 t}F(t)x = 2e^{-\lambda_0 t}\int_0^t S(s, A)\lambda^3\hat{F}(\lambda)x ds - 2e^{-\lambda_0 t}(C(t, A) - I)\lambda\hat{F}(\lambda)x.$$

Using Propositions 6.3.1 and 2.4.1 (iv) and setting $t \to \infty$, we obtain the required result.

Analogously, using Propositions 6.3.1 and 2.4.1 (v), we obtain the following assertion.

Theorem 6.3.3 ([239]). *Let a C_0-cosine operator function $C(\cdot, A)$ satisfy condition (6.2) with $\lambda_0 \in \mathbb{R}_+$. Then for each $\lambda > \lambda_0$ and each $x \in E$, we have*

$$\lim_{t\to\infty} 2e^{-\lambda_0 t}G(t)x = \hat{G}(\lambda)\lambda(\lambda^2/\lambda_0^2 - 1)Px,$$

which is equal to $\frac{1}{\lambda_0^2}(A_c - A)x$ if $x \in \mathcal{N}(\lambda_0^2 I - A)^{-1}$ and is equal to zero if $x \in \overline{\mathcal{R}(\lambda_0^2 I - A)^{-1}}$.

As was mentioned above, if $C(t, A)$ strongly converges as $t \to \infty$, then $C(\cdot, A) \equiv I$, and $F(\cdot)$ and $G(\cdot)$ grow. In what follows, we will consider the behavior of $F(\cdot)$ and $G(\cdot)$ under the assumption that

$$\sup_{t>0}\left\| t^{-2}\int_0^t\int_0^s C(u, A)\, du\, ds\right\| < \infty \quad \text{and} \quad t^{-2}\; C(t, A) \to 0 \tag{6.6}$$

strongly as $t \to \infty$. We need the following assertion.

Proposition 6.3.2 ([258]). *Under assumption* (6.6), *we have*

(i) *the mapping* $P : x \to \lim\limits_{t\to\infty} 2t^{-2} \int\limits_0^t \int\limits_0^s C(u,A)x\,du\,ds$ *is a projection with* $\mathcal{R}(P) = \mathcal{N}(A)$, $\mathcal{N}(P) = \overline{\mathcal{R}(A)}$, *and* $D(P) = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$;

(ii) *there exists* $x := -\lim\limits_{t\to\infty} 2t^{-2} \int\limits_0^t \int\limits_0^s \int\limits_0^u \int\limits_0^v C(\tau,A)y\,d\tau dv du ds$ *iff* $y \in A(D(A) \cap \overline{\mathcal{R}(A)})$ $(= \mathcal{R}(A)$ *in the case where* $C(\cdot,A)$ *is* $(C,2)$*-ergodic, i.e.,* $D(P) = E)$. *Moreover, this element* x *is a unique solution of the equation* $Ax = y$ *in* $\overline{\mathcal{R}(A)}$, *i.e.,* $x = \tilde{A}^{-1}y$, *where* $\tilde{A} = A|_{\overline{\mathcal{R}(A)}}$.

Using Proposition 2.4.1 (iv) and the proposition mentioned above, we obtain the following theorem.

Theorem 6.3.4 ([239]). *Under assumption* (6.6), *the following assertions hold:*

(i) *there exists the limit* $y = \lim\limits_{t\to\infty} 2t^{-2}F(t)x$ *iff* $\hat{F}(\lambda)x \in \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$ *for a certain (and all)* $\lambda > \omega$. *When the limit exists,* $y = \lambda^3 P\hat{F}(\lambda)x$ *and is independent of* λ;

(ii) *for* $\hat{F}(\lambda)x \in \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, $z = \lim\limits_{t\to\infty} 2t^{-2} \int\limits_0^t \int\limits_0^s F(\tau)x\,d\tau ds$ *does exist iff* $\hat{F}(\lambda)x \in A(D(A) \cap \mathcal{R}(A))$ *for a certain (and all)* $\lambda > \omega$. *In this case,* $z = -\lambda(\lambda^2 I - A)\tilde{A}^{-1}\hat{F}(\lambda)x$, *which is independent of* λ.

Proof. We have from (iv) of Proposition 2.4.1 that

$$\frac{2}{t^2}F(t)x = \frac{2}{t^2}(I - C(t,A))\lambda\hat{F}(\lambda)x + \frac{2}{t^2}\int_0^t \int_0^s C(\tau,A)\lambda^3\hat{F}(\lambda)x\,d\tau ds \tag{6.7}$$

and

$$\begin{aligned}\frac{2}{t^2}\int_0^t \int_0^s F(\tau)x d\tau ds &= \frac{2}{t^2}\int_0^t \int_0^s \int_0^u \int_0^v C(\tau,A)\lambda^3\hat{F}(\lambda)x\,d\tau dv du ds \\ &\quad - \frac{2}{t^2}\int_0^t \int_0^s C(\tau,A)\lambda\hat{F}(\lambda)x\,d\tau ds + \lambda\hat{F}(\lambda)x. \end{aligned}\tag{6.8}$$

Then assertions (i) and (ii) follow from (6.7) and (6.8), respectively, as a consequence of Proposition 6.3.2.

By Propositions 2.4.1 (v), 6.3.2, and 2.4.2 (ii), we have the following theorem.

Theorem 6.3.5. *Under assumption* (6.6), *we have the following assertions:*

(i) *if* $x \in \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$, *then* $s\text{-}\lim\limits_{t\to\infty} 2t^{-2}G(t)x = A_c Px$;

(ii) *if* $x \in A(D(A) \cap \mathcal{R}(A))$, *then* $s\text{-}\lim\limits_{t\to\infty} 2t^{-2} \int\limits_0^t \int\limits_0^s G(\tau)x\,d\tau ds = -(A_c - A)\tilde{A}^{-1}x = x - A_c\tilde{A}^{-1}x$, *where* $\tilde{A} = A|_{\overline{\mathcal{R}(A)}}$.

Chapter 7

Ergodic properties of operator C_0-semigroups were considered, e.g., in [17, 20, 66, 76].

7.1. Standard Limits

Proposition 7.1.1 ([146]). *Let $A \in \mathcal{C}(M, 0)$. For any $x = y \dot{+} z \in \mathcal{R}(A) \oplus \mathcal{N}(A)$,*

$$\left\| \frac{1}{T} \int_0^T C(t, A)x dt - z \right\| = O(|T|^{-1}),$$

$$\left\| \frac{1}{T} \int_0^T S(t, A)x dt - \frac{T}{2} z \right\| = O\Big(|T|^{-1}\Big) \quad as \quad T \to \infty.$$

Proposition 7.1.2 ([146]). *In the case where $A \in \mathcal{C}(M, 0)$ and E is reflexive, we have $E = \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$, and, moreover, for each $x \in E$, we have the strong convergence of $\frac{1}{T} \int_0^T C(t, A)x \, dt$ to x as $T \to \infty$.*

The following definition for C_0-cosine operator functions is analogous to Definition 7.1.8 in [17, p. 69] for C_0-semigroups.

Definition 7.1.1. A C_0-semigroup $\exp(\cdot A)$ is said to be *weakly* (*strongly, uniformly*) (C, α) *ergodic at infinity* if the operator $C_t^\alpha[C(\cdot, A)]x := \alpha t^{-\alpha} \int_0^t (t - s)^{\alpha - 1} C(s, A)x \, ds$ does exist for all $t > 0$; $\int_0^\infty e^{-\lambda t} \|C_t^\alpha[C(\cdot, A)]x\| dt < \infty$ for all $x \in E$ and $\lambda > \max(0, \omega(A))$, and if the limit (C, α)-$\lim C(\cdot, A) := \lim_{t \to \infty} C_t^\alpha[C(\cdot, A)]$ exists in the weak (strong, uniform) operator topology. This is the so-called *Cesaro limit.*

Definition 7.1.2. A C_0-cosine operator function $C(\cdot, A)$ is said to be *weakly* (*strongly, uniformly*) *ergodic in the Abel sense* if the limit

$$(A)\text{-} \lim_{t \to \infty} C(t, A) := \lim_{\lambda \to 0+} \lambda \int_0^\infty e^{-\lambda t} C(t, A) dt \equiv \lim_{\lambda \to 0+} \lambda^2 R(\lambda^2, A) \tag{7.1}$$

exists in the corresponding operator topology.

Setting $t \to 0+$ instead of $t \to \infty$ or $\lambda \to \infty$ instead of $\lambda \to 0+$, we obtain the definition of ergodicity at zero.

Theorem 7.1.1 ([76]). *If for a fixed $\alpha \geq 0$, there exists the limit (C, α)-$\lim_{\xi \to \infty} x(\xi) = y \in E$, then for $\beta \geq \alpha$, there exist the limits (C, β)-$\lim_{\xi \to \infty} x(\xi) = A$-$\lim_{\xi \to \infty} x(\xi) = y$.*

Proposition 7.1.3 ([259]). *A C_0-cosine operator function $C(\cdot, A)$ given on the Grothendieck space E is strongly $(C,1)$-ergodic iff the following conditions hold:*

(i) $\|S(t,A)\| = O(t) \quad as \quad t \to \infty$;

(ii) $s\text{-}\lim\limits_{t\to\infty} t^{-1}C(t,A)S(s,A) = 0$ *for all* $s \in \mathbb{R}_+$;

(iii) $w^*\text{-}\mathrm{cl}(\mathcal{R}(A^*)) = \overline{\mathcal{R}(A^*)}$.

Proposition 7.1.4 ([259]). *Under the conditions of Proposition* 7.1.3 *with the space E having the Dunford–Pettis property, a C_0-cosine operator-valued $C(\cdot, A)$ is uniformly $(C,1)$-ergodic iff condition* (i) *of Proposition* 7.1.3 *holds, $\|C(t,A)S(s,A)\| = O(t)$ as $t \to \infty$ for each $s \in \mathbb{R}_+$, and, finally, $w^*\text{-}\mathrm{cl}(\mathcal{R}(A^*)) = \overline{\mathcal{R}(A^*)}$.*

We set $T(t,A) := \int\limits_0^t (t-s)C(s,A)ds$ and define $Q^2_{w^*}$ as on the page 148.

Proposition 7.1.5 ([259]). *Let E be a Grothendieck space, and let $K(t,Q)$ be a w^*-continuous C_0-cosine operator function. If $\|T(t,A)\| = O(t^2)$ as $t \to \infty$ and $w^*\text{-}\lim\limits_{t\to\infty} t^{-2}K(t,Q)T(s)^*x^* = 0$ for all $s \in \mathbb{R}_+$, then $\mathcal{R}(Q^2_{w^*}) = \mathcal{N}(Q)$, $\mathcal{N}(Q^2_{w^*}) = \mathcal{R}(Q)$, and $D(Q^2_{w^*}) = E^*$.*

Moreover, if $s\text{-}\lim\limits_{t\to\infty} t^{-2}K(t,Q)T(s)^* = 0$ for all $s \in \mathbb{R}_+$ then the C_0-cosine operator function $K(\cdot, Q)$ is strongly $(C,2)$-ergodic.

Proposition 7.1.6 ([259]). *A C_0-cosine operator function $C(\cdot, A)$ on the Grothendieck space E is strongly $(C,2)$-ergodic iff*

(i) $\|T(t,A)\| = O(t^2)$ *as* $t \to \infty$;

(ii) $s\text{-}\lim\limits_{t\to\infty} t^{-2}C(t,A)T(s,A) = 0$ *for all* $s \in \mathbb{R}_+$;

(iii) $w^*\text{-}\mathrm{cl}(\mathcal{R}(A^*)) = \overline{\mathcal{R}(A^*)}$.

Proposition 7.1.7 ([259]). *Under the conditions of Proposition* 7.1.6, *let the space E have the Dunford–Pettis property. In this case, a C_0-cosine operator function $C(\cdot, A)$ is uniformly $(C,2)$-ergodic iff $\|T(t)\| = O(t^2)$, $\|C(t,A)T(s,A)\| = O(t^2)$ as $t \to \infty$ and $s \in \mathbb{R}_+$, and also w^*-$\mathrm{cl}(\mathcal{R}(A^*)) = \overline{\mathcal{R}(A^*)}$.*

For any $x \in E$, we set $P_c x := s\text{-}\lim\limits_{t\to\infty} \dfrac{1}{t} S(t,A)x$,

$$P_a x := s\text{-}\lim_{\lambda\to 0+} \lambda \int\limits_0^\infty e^{-\lambda t} C(t,A)x dt, \quad \text{and} \quad P_t x := s\text{-}\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} C(kt,A)x \right).$$

Proposition 7.1.8 ([257])**.** *For $A \in \mathcal{C}(M,0)$, the operators P_c and P_a coincide and are projections. We have the relations*

$$\mathcal{R}(P_c) = \mathcal{N}(A) = \bigcap_{s>0} \mathcal{N}(C(s,A) - I), \quad \mathcal{N}(P_c) = \overline{\mathcal{R}(A)} = \overline{\bigcup_{s>0} \mathcal{R}(C(s,A) - I)},$$

$$D(P_c) = \bigcap_{s>0} \mathcal{N}\Big(C(s,A) - I\Big) \oplus \bigcup_{s>0} \mathcal{R}\Big(C(s,A) - I\Big)$$

$$= \{x \in E : \exists \{t_n\}, t_n \to \infty, \quad w\text{-}\lim_{n\to\infty} (S(t_n,A)x)/t_n \quad \textit{does exist}\}.$$

Proposition 7.1.9 ([257])**.** *Let $A \in \mathcal{C}(M,0)$. For each $t \in \mathbb{R}_+$, the operator P_t is a projection, and*

$$\mathcal{R}(P_t) = \mathcal{N}(C(t,A) - I), \quad \mathcal{N}(P_t) = \overline{\mathcal{R}(C(t,A) - I)},$$

$$D(P_t) = \mathcal{N}(C(t,A) - I) \oplus \overline{\mathcal{R}(C(t,A) - I)}$$

$$= \Big\{x \in E : \exists \{n_k\}, n_k \to \infty, \quad w\text{-}\lim_{k\to\infty} \left(\frac{1}{n_k}\sum_{l=0}^{n_k-1} C(lt,A)x\right) \quad \textit{does exist}\Big\}.$$

Proposition 7.1.10 ([257])**.** *Let $A \in \mathcal{C}(M,0)$. Let there exist $\delta > 0$ such that the operator $C(t,A) + I$ is invertible for $t \in (0,\delta)$ (in particular, this holds if $\|C(t,A) - I\| < 2$ for $t \in (0,\delta)$). Then $P_t = P_c$ for all $t \in (0,2\delta)$.*

Consider the operator

$$H_t^\beta[C(\cdot,A)]x = \frac{\beta}{t^\beta}\int_0^t s^{\beta-1}C(s,A)x\,ds, \quad t > 0,\ x \in X. \tag{7.2}$$

Theorem 7.1.2 ([176])**.** *Let $C(\cdot,A)$ be a bounded C_0-cosine operator function on a Banach space E. The following assertions are equivalent:*

(i) $0 \in C\sigma(A) \cup \rho(A)$;

(ii) *the function $C(\cdot,A)$ is H_t^β-stable for all $\beta > 0$;*

(iii) $\lim_{n\to\infty} H_{t_n}^{\beta_0}[C(\cdot,A)] = 0$ *in the weak operator topology for a certain $\beta_0 > 0$ and a certain positive sequence $\{t_n\}$ converging to ∞.*

It is clear from the proof of this theorem that the boundedness of $C(\cdot,A)$ is not necessary for the implication (iii) $\Longrightarrow$ (i).

It is well known that a generalized solution of the abstract Cauchy problem

$$u''(t) = Au(t), \quad t \in (-\infty,\infty), \qquad u(0) = x, \quad u'(0) = y,$$

is given by the formula $u(t) = C(t,A)x + S(t,A)y$.

Theorem 7.1.3 ([176]). *Let $C(\cdot, A)$ be a bounded C_0-cosine operator function on a Banach space E, and assume that $0 \in C\sigma(A) \cup \rho(A)$. Then a generalized solution $u(t)$ is H_t^β-stable for all $\beta > 0$, for all $x \in E$, and for all y from a certain dense subset of E.*

Theorem 7.1.4 ([257]). *Let $C(\cdot, A)$ be a bounded cosine operator function on a Banach space E, and assume that $0 \in \rho(A) \cup C\sigma(A)$. Then the following conditions are equivalent:*

(i) $y \in A(D(A) \cap \overline{\mathcal{R}(A)})$;

(ii) $x := -\lim\limits_{t\to\infty} 2t^{-2} \int_0^t \int_0^s \int_0^u \int_0^v C(\tau, A)y\, d\tau dv du ds$ *does exist*;

(iii) *for a certain sequence* $\{t_m\}$, *the weak limit* $x := -w\text{-}\lim\limits_{t_m\to\infty} 2t_m^{-2} \int_0^{t_m} \int_0^s \int_0^u \int_0^v C(\tau, A)y\, d\tau dv du ds$ *does exist.*

Such an x is a unique solution of the equation $Ax = y$ in $\overline{\mathcal{R}(A)}$, i.e., $x = (\tilde{A})^{-1}y$, where $\tilde{A} = A|_{\overline{\mathcal{R}(A)}}$.

Theorem 7.1.5 ([176]). *Let $C(\cdot, A)$ be a bounded cosine operator function on a Banach space E. The following conditions are equivalent:*

(i) $0 \in \rho(A) \cup C\sigma(A)$;

(ii) *for all $\beta > 0$, we have* $s\text{-}\lim\limits_{t\to\infty} C_t^\beta[C(\cdot, A)] = 0$;

(iii) *for a certain $\beta_0 > 0$ and a certain positive sequence $\{t_n\}$ converging to ∞ as $n \to \infty$, we have* $w\text{-}\lim\limits_{n\to\infty} C_{t_n}^{\beta_0}[C(\cdot, A)] = 0.$

The assertion that $\lim\limits_{t\to\infty} C_t^\beta[u(\cdot)] = 0$, which is similar to Theorem 7.1.3 (for a generalized solution $u(\cdot)$ of a second order equation) can be proved in the same way.

7.2. Tauberian Theorem

As was noted, for a C_0-cosine operator function $C(t, A)$, the notion of stability is vacuous because the convergence $C(t, A) \to P \in B(E)$ as $t \to \infty$ implies $C(t, A) \equiv I$. It is clear that integrated semigroups or cosine functions do not have asymptotic convergence properties because they naturally increase polynomially [116].

On the other hand, one can consider the Cesaro averaging of cosine operator functions. The necessary and sufficient condition for a bounded C_0-cosine operator function to be (C, β)-stable for any $\beta > 0$, i.e., $1/t^\beta \int_0^t (t - s)^{\beta-1} C(s, A) ds \to 0$ strongly as $t \to \infty$, is that $0 \in \rho(A) \cup C\sigma(A)$, as was seen in the previous section. Therefore, the behavior of (C, β)-averages is defined just by the point 0. It is known that for at least polynomially bounded C_0-semigroups, the behavior of some Cesaro type averages is determined by the behavior of the resolvent in a neighborhood of zero [168]. We are going to show here that for C_0-cosine operator functions, the situation is very similar.

To see that the behavior of Cesaro type averages is closely connected with the behavior of the resolvent of a polynomially bounded cosine operator function in a neighborhood of zero, we consider the basic example of $n \times n$ nilpotent matrix

$$Q = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ 1 & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $Q^n = 0$ and $\cosh(t\sqrt{Q}) = I + Q\frac{t^2}{2!} + \cdots + Q^{n-1}\frac{t^{2(n-1)}}{(2n-2)!}$; therefore, the C_0-cosine operator function $\cosh(t\sqrt{Q})$, $t \geq 0$, is certainly polynomially bounded. The resolvent of Q is given by

$$(\lambda^2 I - Q)^{-1} = \lambda^{-2}\left(I - \frac{Q}{\lambda^2}\right)^{-1} = \lambda^{-2}\left(I + \frac{Q}{\lambda^2} + \frac{Q^2}{\lambda^4} + \cdots + \frac{Q^{n-1}}{\lambda^{2(n-1)}}\right).$$

Hence, for $\alpha = 2n-2$, we have

$$\lim_{\lambda \to 0+} \lambda^{2+\alpha}(\lambda^2 I - Q)^{-1} = Q^{n-1}$$

and $\|\lambda^{2+\alpha}(\lambda^2 I - Q)^{-1}\| \leq n$ for all $|\lambda| \leq 1$.

From the point of view of Cesaro convergence, we have

$$\lim_{t\to\infty} \frac{1}{t^{\alpha+1}} \int_0^t \cosh(s\sqrt{Q})\, ds = \frac{Q^{n-1}}{(2n-1)!},$$

and, more generally,

$$\lim_{t\to\infty} \frac{1}{t^{\alpha+m}} \int_0^t (t-s)^{m-1} \cosh(s\sqrt{Q})\, ds = \frac{\Gamma(m)}{\Gamma(\alpha+m+1)} Q^{n-1} \tag{7.3}$$

for all $m = 1, 2, \ldots$.

Let A be the generator of a polynomially bounded cosine operator function acting on a Banach space E, i.e., there exist numbers $M > 0$ and $\beta \geq 0$ such that

$$\|C(t, A)\| \leq M(1+t)^\beta \quad \text{for all } t \geq 0. \tag{7.4}$$

Let $P \in B(E)$ be a bounded linear operator.

Theorem 7.2.1 ([169]). *Let $\alpha > 0$, and let (7.4) hold. Then the conditions*

(i) $\lambda^{\alpha+2}(\lambda^2 I - A)^{-1} \to P$ *in the strong operator topology as* $\lambda \to 0+$ *in* $\mathbb{R}$;

(ii) *there exist* $C > 0, N \geq 0$ *and* $\rho_0 > 0$ *such that*

$$\|\rho^{2+\alpha}(\rho^2 e^{i2\varphi} - A)^{-1}\| \leq \frac{C}{\cos^N(\varphi)}, \quad 0 < \rho \leq \rho_0, \ \varphi \in (-\pi/2, \pi/2),$$

are necessary and sufficient for the existence of a positive integer m *such that*

$$\lim_{t\to\infty} \frac{\Gamma(m+\alpha+1)}{\Gamma(m)t^{m+\alpha}} \int_0^t (t-s)^{m-1} C(s,A)x\,ds = Px \tag{7.5}$$

for each $x \in E$.

Remark 7.2.1. Since for an operator A that generates a polynomially bounded distribution cosine, we can find the corresponding polynomially bounded and strongly continuous m-times integrated cosine [80, 214, 215], the same theorem ought to be valid for the distribution case.

Remark 7.2.2. Assume that $\lambda^2(\lambda^2 I - A)^{-1} \to P$ as $\lambda \to 0+$, i.e., (i) holds with $\alpha = 0$. Then we obtain from the Hilbert identity that

$$\lambda^2\mu^2(\lambda^2 I - A)^{-1}(\mu^2 I - A)^{-1} = \frac{\lambda^2\mu^2}{\lambda^2 - \mu^2}((\mu^2 I - A)^{-1} - (\lambda^2 I - A)^{-1}). \tag{7.6}$$

Now setting $\lambda = \sqrt{2}\mu$ and then passing to the limit as $\mu \to 0+$, we obtain from (7.6) that $P^2 = P$, i.e., P is a projection. In the case where $\alpha > 0$ in (i), the operator P is no longer a projection. It follows from (7.6) that P has the property $P^2 = 0$.

Chapter 8

UNIFORMLY BOUNDED C_0-COSINE OPERATOR FUNCTIONS

As in the case of operator semigroups, the norm-continuity is a very restrictive requirement for cosine operator functions, since it implies the boundedness of the generator. Conditions for generation of a C_0-cosine operator function for an infinitesimal operator A are more restrictive than those for generation of operator semigroups.

8.1. Norm-Continuity

It is very natural that the boundedness of A in the case of a C_0-cosine operator function follows under weaker additional assumptions than in the case of operator semigroups. So, for example, the condition $\|tA\exp(-tA)\| \leq C$ implies the boundedness of A for $C = 1/e$ (see [17]), and in the case of cosine operator functions, for boundedness of A, the boundedness $\|AS(t,A)\| \leq \text{const}$ with any constant is sufficient.

Definition 8.1.1. A C_0-cosine operator function $C(\cdot, A)$ is continuous in the uniform operator topology (*norm-continuous*) if the function $C(\cdot, A) : \mathbb{R} \to B(E)$ is continuous in the operator norm.

Proposition 8.1.1 ([273]). *Let a C_0-cosine operator function $C(\cdot, A)$ be continuous in the uniform operator topology. Then $A \in B(E)$ and*

$$C(t, A) = \sum_{k=0}^{\infty} t^{2k} A^k/(2k)!, \quad t \in \mathbb{R}, \tag{8.1}$$

and, moreover, the series uniformly converges in t on each finite closed interval $[0, T]$.

Sometimes, in the literature, series (8.1) is written as $\cosh(t\sqrt{A})$ analogously to the scalar case.

We note that a C_0-sine operator function $S(\cdot, A)$ is always uniformly continuous in $t \in \mathbb{R}$, as follows from its definition.

Proposition 8.1.2 ([272]). *Let $A \in B(E)$. Then series (8.1) is a C_0-cosine operator function whose generator is A.*

Theorem 8.1.1 ([87]). *Each of the following conditions is equivalent to the norm-continuity of $C(\cdot, A)$:*

(i) $\lim_{t\to 0} \|C(t, A) - I\| = 0$;

(ii) $\lim_{t\to 0} \|t^{-1}S(t, A) - I\| = 0$;

(iii) *the generator A is bounded*;

(iv) $\mathcal{R}(C(t, A)) \subseteq E^1$ *for all $t \in (\alpha, \beta)$ with certain $\alpha < \beta$*;

(v) *the inclusion $\mathcal{R}(S(t, A)) \subseteq D(A)$ and the strong continuity of the function $t \to AS(t, A)$ hold for all $t \in (\alpha, \beta)$ with certain $\alpha < \beta$.*

Proposition 8.1.3 ([7]). *The generator A of a C_0-cosine operator function with a non-quasi-analytic weight χ is bounded iff one of the following conditions holds*:

(i) *for a certain $\varepsilon > 0$, we have* $\sup_{0<t<\varepsilon} \|C(t, A) - I\| < 1$;

(ii) *the C_0-cosine operator function $C(\cdot, A)$ is the restriction to $\mathbb{R}$ of an entire operator function $\tilde{C}(\cdot) : \mathbb{C} \to B(E)$ of exponential type (equal to $r(A)$).*

Proposition 8.1.4 ([273]). *Let a C_0-cosine operator function $C(\cdot, A)$ be twice strongly differentiable at zero. Then $A \in B(E)$.*

Proposition 8.1.5 ([206]). *Let $A \in \mathcal{C}(M, 0)$. Then $A \in B(E)$ iff the spectrum $\sigma(A)$ is bounded.*

Proposition 8.1.6. *Let a C_0-cosine operator function $C(\cdot, A)$ be norm-continuous. Then*

(i) $\lim_{t\to 0} \left\| \frac{2}{t^2} \int_0^t S(s, A)ds - I \right\| = 0$;

(ii) *for sufficiently small h, the operator $\int_0^h S(s, A)ds$ has a bounded inverse*;

(iii) *for sufficiently small* h, *we have the relation*

$$A = (C(h, A) - I)\left(\int_0^h S(s, A)ds\right)^{-1}.$$

Definition 8.1.2. A *Grothendieck space* is a Banach space in which every w^*-convergent sequence in E^* is w-convergent.

Definition 8.1.3. We say that a Banach space E has the *Dunford–Pettis property* if $\langle x_n, x_n^* \rangle \to 0$ whenever x_n weakly converges to zero in E and x_n^* weakly converges to zero in E^*.

Proposition 8.1.7 ([259]). *Any* C_0*-cosine operator function* $C(\cdot, A)$ *given on a Grothendieck space with the Dunford–Pettis property (for example,* $E = L^\infty$ *is such a space) is norm-continuous, i.e.,* $A \in B(E)$.

Before formulating the next assertion, we recall [147] that if E is an H.I. space and $B \in B(E)$, then there exists a unique point $\lambda_B \in \sigma(B)$ such that the operator $B - \lambda_B I$ is strictly singular. Moreover, $B - \lambda_B I$ is a Riesz operator.

Proposition 8.1.8 ([248]). *Let* E *be an H.I. Banach space, and let* $C(\cdot, A)$ *satisfy condition* (7.4). *Then* $A \in B(E)$, *and there is a positive integer* m *such that* $(A - \lambda_A I)^m$ *is a compact operator.*

Proposition 8.1.9 ([7]). *If* $B \in B(E)$ *and a* C_0*-cosine operator function* $C(\cdot, B)$ *is uniformly bounded in* $t \in \mathbb{R}$, *then the following Bernshtein inequality holds*:

$$\|B\| \le r(B) \cdot \sup_{t \in \mathbb{R}} \|C(t, B)\|,$$

where $r(B)$ *is the spectral radius of the operator* B.

Proposition 8.1.10 ([27, 242]). *Let* $B \in B(E)$. *Then the functions* $C(t, -B^2)x$ *and* $S(t, -B^2)y$ *are uniformly bounded in* $t \in \mathbb{R}$ *for any* $x, y \in E$ *iff there exists an equivalent norm* $\|\cdot\|_*$ *on* E *such that* $\|\exp(itB)\|_* \le 1$ *for* $t \in \mathbb{R}$ *(such operators* B *are said to be Hermitian-equivalent on* E*).*

Proposition 8.1.11 ([27, 242]). *An operator* $B \in B(E)$ *is Hermitian-equivalent on* E *iff there exists a constant* $C > 0$ *such that*

$$\|\sin(tB)\| \le C, \quad t \in \mathbb{R}.$$

Proposition 8.1.12 ([86]). *In a Banach algebra $\mathcal{B}$ with unit, let a C_0-cosine operator function $C(\cdot, A)$, $A \in \mathcal{B}$ be given. Then for $\nu^2 > \sup\limits_{\lambda \in \sigma(A)} (|\lambda| + \operatorname{Re}\lambda)/2$, we have*

$$C(t, A) = \frac{1}{2\pi i} \int\limits_{\nu - i\infty}^{\nu + i\infty} e^{\lambda t} \lambda R(\lambda^2, A) d\lambda, \quad t \in \overline{\mathbb{R}}_+,$$

$$S(t, A) = \frac{1}{2\pi i} \int\limits_{\nu - i\infty}^{\nu + i\infty} e^{\lambda t} R(\lambda^2, A) d\lambda, \quad t \in \overline{\mathbb{R}}_+.$$

Proposition 8.1.13 ([100]). *If for a C_0-cosine operator function $\mathcal{R}(S(t, A)) \subseteq D(A)$ for $t \in \mathbb{R}$ and $\|AS(t, A)\| \leq \mathrm{const}$ for $t \in [a, b]$, $a < b$, then $A \in B(E)$.*

Proposition 8.1.14 ([100]). *If for a C_0-cosine operator function $\mathrm{SV}(C(\cdot, A), t) \leq \mathrm{const}$ for a certain $t \in \mathbb{R}_+$, then $A \in B(E)$.*

8.2. Positivity of Perturbation Families

Definition 8.2.1. In the case where E is a Banach lattice with a positive cone E^+, we say that a function $\mathcal{L}(\cdot)$ is *positive* on E if for each $t \in \overline{\mathbb{R}_+}$, the operator $\mathcal{L}(t)$ is positive (we write $\mathcal{L}(t) \succeq 0$) in the sense that $\mathcal{L}(t)E^+ \subseteq E^+$.

In the case where E is a Hilbert space with the inner product $(\cdot, \cdot)$, we say that $\mathcal{L}(\cdot)$ is positive (we write $\mathcal{L}(t) \geq 0$) if for each $t \in \overline{\mathbb{R}}_+$, the operator $\mathcal{L}(t)$ is positive in the sense that $(\mathcal{L}(t)x, x) \geq 0$ for all $x \in E$.

Proposition 8.2.1 ([197]). *A C_0-cosine operator function $C(\cdot, A)$ dominates I, i.e., $C(\cdot, A) - I$ is positive in the sense of a Banach lattice or in the sense of a Hilbert space iff the generator A is bounded and positive.*

The following propositions are a reformulation of properties of a C_0-family of multiplicative perturbations and a C_0-family of additive perturbations.

Let $F_B^\mu(\cdot)$ and $G_B^\mu(\cdot)$ be functions defined for $B \in B(E)$ by the formulas

$$\begin{aligned} F_B^\mu(t)x &:= (A - \mu I) \int_0^t S(s, A)Bx\, ds, \quad x \in E,\ t \in \overline{\mathbb{R}}_+, \\ G_B^\mu(t)x &:= B(A - \mu I) \int_0^t S(s, A)x\, ds, \quad x \in E,\ t \in \overline{\mathbb{R}}_+. \end{aligned} \tag{8.2}$$

Then $F_B^\mu(\cdot)$ is a C_0-family of multiplicative perturbations and $G_B^\mu(\cdot)$ is a C_0-family of additive perturbations.

Proposition 8.2.2 ([239]). *Let E be a Banach lattice. Each C_0-family of multiplicative perturbations $F_B^{\mu}(\cdot)$ for a C_0-cosine operator function $C(\cdot, A)$ on E defined in* (8.2) *with $\mu \leq 0$ and $B \succeq 0$ is positive iff the operator A is positive. The same holds for a C_0-family of additive perturbations.*

Proposition 8.2.3 ([239]). *Let E be a Hilbert space. Each C_0-family of multiplicative perturbations $F_B^{\mu}(\cdot)$ for a C_0-cosine operator function $C(\cdot, A)$ on E defined in* (8.2) *with $\mu \leq 0$ and $B \geq 0$ that commutes with $C(\cdot, A)$ is positive iff the operator A is positive. The same holds for a C_0-family of additive perturbations.*

Chapter 9

ALMOST-PERIODIC C_0-COSINE OPERATOR FUNCTIONS

Let us recall in brief the definition of almost periodicity of operator functions.

Definition 9.0.1. A function $f(\cdot) : \overline{\mathbb{R}}_+ \to E$ is said to be *almost-periodic* if for each $\epsilon > 0$, the set $J(f, \epsilon) = \{\tau > 0 : \|f(t + \tau) - f(t)\| \leq \epsilon$ for all $t \in \overline{\mathbb{R}}_+\}$ is relatively dense in $\overline{\mathbb{R}}_+$. That is, there exists $l \in \mathbb{R}_+$ such that each subinterval from $\overline{\mathbb{R}}_+$ of length l intersects $J(f, \epsilon)$. An operator function $Q(\cdot) : \overline{\mathbb{R}}_+ \to B(E)$ is said to be *almost-periodic* if for each $x \in E$, the function $Q(\cdot)x$ is almost periodic.

9.1. Almost Periodicity of the Basic Families

Definition 9.1.1. A C_0-cosine operator function or a C_0-sine operator function are said to be *almost-periodic* (a.-p.) or *uniformly a.-p.* if for any $x \in E$, the functions $C(\cdot, A)x$ or $S(\cdot, A)x$ are a.-p. (uniformly a.-p.).

Proposition 9.1.1 ([103]). *If E is weakly sequentially complete, then a weakly a.-p. C_0-cosine operator function is almost-periodic.*

Theorem 9.1.1 ([62]). *A C_0-cosine operator function $C(\cdot, A)$ is almost-periodic iff the following three conditions hold:*

(i) *the C_0-cosine operator function $C(\cdot, A)$ is uniformly bounded;*

(ii) *the spectrum $\sigma(A) \subseteq \mathbb{R}_-$;*

(iii) *the set of eigenvectors of the generator A is total on the space E.*

If, moreover, $\mu \in \sigma(A)$ is an isolated point of the spectrum, then μ is a simple pole of the resolvent $(\lambda I - A)^{-1}$ and $E = \overline{\mathcal{R}(\mu I - A)} \oplus \mathcal{N}(\mu I - A)$.

Theorem 9.1.2 ([62]). *The Cauchy problem* (3.1) *has an a.-p. generalized solution for any $u^0, u^1 \in E$ iff conditions* (i)–(iii) *of Theorem* 9.1.1 *hold and $0 \in \rho(A)$.*

Theorem 9.1.3 ([62]). *A C_0-cosine operator function $C(\cdot, A)$ and a C_0-sine operator function $S(\cdot, A)$ are uniformly a.-p. iff the following three conditions hold:*

(i) *the C_0-cosine operator function $C(\cdot, A)$ and the C_0-sine operator function $S(\cdot, A)$ are uniformly bounded in $t \in \mathbb{R}$;*

(ii) *the set $\sigma(A)$ is a harmonic subset in $\mathbb{R}_-$ and $0 \in \rho(A)$;*

(iii) *the set of eigenvectors of the generator A is total in the space E.*

Proposition 9.1.2 ([62]). *If a C_0-cosine operator function $C(\cdot, A)$ is uniformly a.-p., then $\sigma(A)$ consists of simple poles of the resolvent $(\lambda I - A)^{-1}$. In this case, $\sigma(A) = P\sigma(A)$.*

Proposition 9.1.3 ([161]). *The following conditions are equivalent:*

(i) *a C_0-cosine operator function $C(\cdot, A)$ is periodic as an operator function;*

(ii) *the C_0-cosine operator function $C(\cdot, A)$ is strongly periodic;*

(iii) *the C_0-cosine operator function $C(\cdot, A)$ is weakly periodic.*

Theorem 9.1.4 ([141, 205]). *A uniformly bounded C_0-cosine operator function $C(\cdot, A)$ is periodic with period 2π iff the following three conditions hold:*

(i) *the spectrum $\sigma(A) \subseteq \{l : l = -k^2, k \in \mathbb{Z}\}$;*

(ii) *the spectrum $\sigma(A)$ consists of simple poles of the resolvent;*

(iii) *the set of eigenvectors of the generator A is total in the space E.*

Under conditions (i)–(iii), *the Riesz projections are given by the formulas*

$$P(-k^2)x = \begin{cases} \dfrac{1}{\pi} \int\limits_0^{2\pi} \cos(ks) C(s, A) x\, ds & \text{for} \quad k \neq 0, \\ \dfrac{1}{2\pi} \int\limits_0^{2\pi} C(s, A) x\, ds & \text{for} \quad k = 0 \end{cases}$$

and, moreover, for $x \in D(A)$, we have the relation

$$C(t, A)x = \sum_{k=0}^{\infty} \cos(kt) P(-k^2) x, \tag{9.1}$$

where the series converges uniformly in $t \in \mathbb{R}$.

Proposition 9.1.4 ([141]). *In the case where $E = H$ and $C(\cdot, A) - 2$ is π-periodic, relation* (9.1) *holds for all $x \in E$ and the convergence of the series is uniform in $t \in \mathbb{R}$.*

Theorem 9.1.5 ([63]). *The function $C(t, A)u^0 + S(t, A)u^1$ is 2π-periodic for any $u^0, u^1 \in E$ iff conditions* (i)–(iii) *of Theorem* 9.1.4 *hold and $0 \in \rho(A)$.*

Proposition 9.1.5 ([141]). *A C_0-cosine operator function $C(\cdot, A)$ is periodic with period T iff the function $F(z) := (1 - e^{-Tz})zR(z^2, A)$ can be analytically continued up to an entire function $\tilde{F}(z)$ such that the following estimate holds for $|z| > r$:*

$$\|\tilde{F}(z)\| \leq Me^{(q|z|^{2-\varepsilon})}, \quad \textit{where} \quad q, M, r, \varepsilon > 0. \tag{9.2}$$

Proposition 9.1.6 ([221]). *The uniformly well-posed Cauchy problem* (3.1) *has periodic solutions with period T iff $A \in \mathcal{C}(M, 0)$ and the function $F(z)/z$ can be analytically continued up to an entire function $Q(z)$ such that estimate* (9.2) *for $Q(z)$ holds for $|z| > r$.*

Proposition 9.1.7 ([139]). *Let a C_0-cosine operator function $C(\cdot, A)$ be given on a Hilbert space H, and let $C(\cdot, A)$ be weakly a.-p. Then we have the relation $C(t, A) = Q^{-1}C(t, V)Q$, where V is a self-adjoint operator given by $V := \sum\limits_{\lambda \geq 0} \lambda P(\lambda)$ and $P(\lambda)$ is a set of mutually orthogonal projections.*

Proposition 9.1.8 ([205]). *The periodicity of a C_0-cosine operator function $C(\cdot, A)x$ for each $x \in D(A)$ implies the periodicity of $C(\cdot, A)$.*

Proposition 9.1.9 ([7]). *Let $\sqrt{\sigma(-A)} \cap \mathbb{R}_+$ be a not more than countable set. Then all solutions of problem* (3.1) *are almost-periodic iff the following conditions hold:*

(i) *the C_0-cosine operator function $C(\cdot, A)$ is uniformly bounded in $t \in \mathbb{R}$;*

(ii) $0 \in \rho(A)$;

(iii) *for each limit point $\lambda_0 \in \sqrt{\sigma(A)}$, there is a sequence $\varepsilon_n \in \mathbb{R}$ converging to zero as $n \to \infty$ such that $s\text{-}\lim\limits_{n\to\infty} \varepsilon_n(\varepsilon_n + i\lambda_0)((\varepsilon_n + i\lambda_0) \cdot I - A)^{-1}x = 0$ for each $x \in E$.*

Proposition 9.1.10 ([7]). *A C_0-cosine operator function $C(\cdot, A)$ is a.-p. in the uniform operator topology iff it is uniformly bounded on $\mathbb{R}$ and $\sqrt{\sigma(A)}$ is a harmonic subset of $i\mathbb{R}$.*

Proposition 9.1.11 ([7]). *Let $A \in \mathcal{C}(M, \omega)$, and let $\sqrt{\sigma(-A)}$ have no limit points in $\mathbb{R}_+$. Then:*

(i) *the linear span of eigenvectors and root vectors of A is dense in E if there exists a function $\chi(t)$ such that*

$$\|C(t, A)\| \leq \chi(t) \quad \textit{and} \quad \chi(t) \leq C(1 + |t|)^\gamma \quad \textit{for} \quad t \in \mathbb{R}, \gamma \geq 0; \tag{9.3}$$

(ii) *under the condition $\lim\limits_{t\to\infty} \chi(t)/t = 0$, where $\chi(\cdot)$ is the function from* (9.3), *the C_0-cosine operator function $C(\cdot, A)$ is periodic with period 1 iff $\sigma(A) \subseteq \{-(2\pi k)^2 : k \in \mathbb{N}\}$.*

In [160], asymptotic almost-periodic in the sense of Stepanov operator semigroups and cosine operator functions were considered.

Proposition 9.2.1. *If a continuous function $f(\cdot) : \overline{\mathbb{R}}_+ \to E$ converges to a certain element $\varphi \in E$ as $t \to \infty$, then*

$$2t^{-2} \int_0^t sf(s)\, ds \ \to \varphi \text{ as } t \to \infty. \tag{9.4}$$

Proof. It is clear that as in the case of Lemma 6.3.1 (see also [17, Lemma 7.3.1]), it suffices to consider the case $\varphi = 0$. We set $t = \tau + \zeta$ and write

$$\frac{2}{t^2} \int_0^t sf(s)\, ds = \frac{2}{(\tau+\zeta)^2} \int_0^\tau sf(s)\, ds + \frac{2}{(\tau+\zeta)^2} \int_\tau^{\tau+\zeta} sf(s)\, ds. \tag{9.5}$$

Since

$$\left\| \frac{2}{(\tau+\zeta)^2} \int_\tau^{\tau+\zeta} sf(s) ds \right\| \leq \sup_{t \geq \tau} \|f(t)\|$$

for all ζ and τ and $f(t) \to 0$ as $t \to \infty$, we can choose τ so large that the second term in (9.5) becomes less than a certain $\varepsilon > 0$. Then we can choose ζ so large that the first term in (9.5) also becomes less than ϵ. This proves (9.4) with $\varphi = 0$.

The next theorem yields necessary and sufficient conditions for each C_0-family of multiplicative perturbations (or each C_0-family of additive perturbations) to be almost-periodic.

Theorem 9.2.1 ([239]). *Each C_0-family of multiplicative perturbations $F(\cdot)$ for $C(\cdot, A)$ is almost periodic iff $C(\cdot, A)$ is almost periodic and $0 \in \rho(A)$. The same assertion holds for a C_0-family of additive perturbations.*

Proof. Let $C(\cdot, A)$ be almost-periodic. Then the condition $0 \in \rho(A)$ implies the almost-periodicity of the function

$$\int_0^t S(s, A)\, ds = \int_0^t S(s, A) A A^{-1} ds = (C(t, A) - I) A^{-1}, \quad t \in \mathbb{R}.$$

Therefore, Proposition 2.4.1 (iv) implies the almost periodicity of $F(\cdot)$.

Conversely, if each C_0-family of multiplicative perturbations is almost-periodic, then two particular C_0-families of multiplicative perturbations $C(t, A) - I$ and $\int\limits_0^t S(s, A) ds$ are almost-periodic functions. If $x \in \mathcal{N}(A)$, then $x = C(s, A)x - \int\limits_0^s S(u, A) Ax du = C(s, A)x$ for all $s \in \overline{\mathbb{R}}_+$ and $x = 2t^{-2} \int\limits_0^t S(s, A) x ds \to 0$ as $t \to \infty$, since an almost-periodic function is bounded. Therefore, A is injective. Then since an almost-periodic function is ergodic (see, e.g., [78, p. 21]), the limit $\frac{1}{s} \int\limits_0^s \int\limits_0^u S(v, A) x\, dv du$ does exist as $s \to \infty$ for each $x \in E$. By Proposition 9.2.1, the limit

$$\frac{2}{t^2} \int_0^t s \frac{1}{s} \int_0^s \int_0^u \int_0^v C(\tau, A) x\, d\tau\, dv\, du\, ds$$

exists as $t \to \infty$ for any $x \in E$. Since $C(\cdot, A)$ is uniformly bounded, we have from Proposition 6.3.2 (i.e., [240, Theorem 3.7]) that $\mathcal{R}(A) = E$. Therefore, $0 \in \rho(A)$.

Remark 9.2.1 ([62]). The assumptions that $C(\cdot, A)$ is almost periodic and $0 \in \rho(A)$ are equivalent to the condition that each mild solution of the Cauchy problem (3.1) is almost-periodic.

We can deduce the following theorem from Theorem 9.2.1.

Theorem 9.2.2. *Each C_0-family of multiplicative perturbations $F(\cdot)$ for $C(\cdot, A)$ is periodic iff $C(\cdot, A)$ is periodic and $0 \in \rho(A)$. In this case, $F(\cdot)$ and $C(\cdot, A)$ have the same period. The same assertion holds for a C_0-family of additive perturbations.*

Chapter 10

COMPACTNESS IN THE THEORY OF C_0-COSINE OPERATOR FUNCTIONS

Compactness properties are widely used in various aspects of the theory of resolving families. We denote by $B_0(E)$ (or $B_0(E, F)$ in the case of distinct spaces) the set of compact operators acting on E.

10.1. Compact Basic Families

Definition 10.1.1. A C_0-cosine operator function $C(\cdot, A)$ is said to be *compact* (we write $C(\cdot, A) \in B_0(E)$) if the operator $C(t, A) \in B_0(E)$ for any $t \in \mathbb{R}_+$. A C_0-sine operator function $S(\cdot, A)$ is said to be *compact* if the operator $S(t, A) \in B_0(E)$ for any $t \in \mathbb{R}$.

Proposition 10.1.1 ([273]). *If an operator $C(t, A) \in B_0(E)$ for each $t \in (\alpha, \beta)$ for certain $\alpha < \beta$, then the C_0-cosine operator function $C(\cdot, A) \in B_0(E)$ and the C_0-sine operator function $S(\cdot, A) \in B_0(E)$.*

Proposition 10.1.2 ([273]). *If an operator $S(t, A) \in B_0(E)$ for each $t \in (\alpha, \beta)$ and for certain $\alpha < \beta$, then the C_0-sine operator function $S(\cdot, A) \in B_0(E)$, $t \in \mathbb{R}$.*

Proposition 10.1.3 ([64]). *If $\dim E = \infty$, then for no $t_0 > 0$, the operators $C(t_0, A)$ and $C(2t_0, A)$ can be compact simultaneously.*

Proposition 10.1.4 ([273]). *Under the condition of Proposition 10.1.1, we necessarily have $\dim E < \infty$.*

Theorem 10.1.1 ([273]). *The following conditions are equivalent:*

(i) *a C_0-sine operator function $S(\cdot, A) \in B_0(E)$;*

(ii) *the resolvent $(\lambda^2 I - A)^{-1} \in B_0(E)$ for any λ with $\operatorname{Re} \lambda > \omega_c(A)$.*

Theorem 10.1.2 ([64]). *The following conditions are equivalent:*

(i) *a generator* $A \in B_0(E)$;

(ii) *the operator* $\lambda^2(\lambda^2 I - A)^{-1} - I \in B_0(E)$ *for each* $\lambda > \omega_c(A)$;

(iii) *the operator* $S(t, A) - tI \in B_0(E)$ *for any* $t \in \mathbb{R}$;

(iv) *the operator* $C(t, A) - I \in B_0(E)$ *for any* $t \in \mathbb{R}$.

Proposition 10.1.5 ([159]). *Let* $C(t, A) - I \in B_0(E)$ *for any* $t \in \mathbb{R}$. *Then* $\lambda(\lambda I - A)^{-1} - \mu(\mu I - A)^{-1} \in B_0(E)$ *for all* $\lambda, \mu \in \rho(A)$ *such that* $\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega_c(A)$.

Proposition 10.1.6 ([159]). *Let* $C(t, A) - I \in B_0(E)$ *for each* $t \in (\alpha, \beta)$ *and for certain* $\alpha < \beta$. *Then* $C(t, A) - I \in B_0(E)$ *for any* $t \in \mathbb{R}$.

Proposition 10.1.7 ([159]). *If* $S(t, A) - tI \in B_0(E)$ *for each* $t \in (\alpha, \beta)$ *and for certain* $\alpha < \beta$, *then* $S(t, A) - tI \in B_0(E)$ *for any* $t \in \mathbb{R}$.

Proposition 10.1.8 ([61]). *Let a* C_0*-sine operator function* $S(t, A)$ *and the operator function* $C(t, A) - I$ *be compact for each* $t \in \mathbb{R}$. *Then the space* E *is necessarily finite-dimensional.*

Proof. By our assumptions, it follows from Theorems 10.1.1 and 10.1.2 that the resolvent $(\lambda I - A)^{-1}$ and the operator $\lambda(\lambda I - A)^{-1} - I$ are compact for certain $\lambda \neq 0$. This implies that I is a compact operator, i.e., E is finite-dimensional.

Let us define the sets

$$NBI_0 = \{t > 0 : \text{the operator } C(t, A) \text{ has no bounded inverse}\},$$

$$NBI_1 = \{t > 0 : \text{the operator } S(t, A) \text{ has no bounded inverse}\}.$$

Proposition 10.1.9 ([159]). *Let* $C(t, A) - I \in B_0(E)$ *for all* $t \in \mathbb{R}$. *Then the sets* NBI_0 *and* NBI_1 *either are empty, simultaneously, or are infinite of continuum cardinality, and there exist constants* $\alpha_0, \alpha_1 > 0$ *such that* $NBI_j \subseteq (\alpha_j, \infty), j = 0, 1$.

10.2. Compactness of the Difference of Cosines

Many linear distributed parameter control systems can be reduced to the form

$$v'(t) = Av(t) + Bu(t),\ v(0) = v_0,\ t \in \mathbb{R}_+, \tag{10.1}$$

where A generating a C_0-semigroup on the state Hilbert or Banach space E and B is the control operator acting from control space to the state space. When we design a feedback control $u(t) = Fv(t)$ for some feedback operator from the state space to the control space, the closed-loop system becomes

$$v'(t) = (A + BF)v(t),\ v(0) = v_0,\ t \in \mathbb{R}_+. \tag{10.2}$$

In the context of the stabilization theory, we want to choose a feedback operator F in order to force the closed-loop system to possess stability properties that are not enjoyed by the original system. One important class in physical applications is that of operators F such that BF is compact on the state space. When BF is compact, it was first proved in [290] that the difference of the semigroups $\exp(A+BF)t$ and $\exp(tA)$ is compact for any positive t. Hence

$$E\omega(A) = E\omega(A + BF), \tag{10.3}$$

where $E\omega$ stands for the essential growth rate of the associated semigroup. Property (10.3) holds for any two C_0-semigroups whenever their difference is compact for some $t > 0$ (see Theorem 3.52 in [202]). This is the basis of the compactness method that was used in studying the stabilization of elastic systems (see [247]) and the spectral property of the transport equation (see [283]). The compactness method was first formulated in [276] for Hilbert spaces, and later on, it was generalized to Banach spaces in [151]; it says that a compact perturbation cannot make the system exponentially stable if it is asymptotically but not exponentially stable.

This leads to the general study of necessary and sufficient conditions for compactness of the difference of two C_0-semigroups. A recent result in [190] says that $\exp(tA) - \exp(tB)$ is compact for some $t > 0$ iff $R(\lambda; A) - R(\lambda; B)$ is compact under the norm-continuity assumption.

On the other hand, it is more convenient to write most of control hyperbolic systems in the form of a second-order system instead of a first-order evolution equation in an abstract space (see [20, 251]):

$$v''(t) = Av(t) + Bu(t),\ v(0) = v_0,\ v'(0) = v_1,\ t \in \mathbb{R}_+. \tag{10.4}$$

System (10.4) can be transferred into the first order equation (3.5); however, there are some problems, since $\mathcal{A}$ does not, in general, generate a C_0-semigroup on $E \times E$. In this connection, the problem of compactness of the difference of two C_0-cosine operator functions is of interest. Let $C(t, A)$ and $C(t, B)$ be two cosine functions on a Banach space E, satisfying $\|C(t, A)\|$, $\|C(t, B)\| \leq Me^{w|t|}$, $t \in \mathbb{R}$, for some constants $M, w \geq 0$. Denote $\Delta_{A,B}(t) = C(t, A) - C(t, B)$ for all $t \in \mathbb{R}$.

Theorem 10.2.1 ([191]). *Let $\Delta_{A,B}(t)$ be norm-continuous for $t > 0$. Then for all $\lambda > w^2$, the operator $R(\lambda; A) - R(\lambda; B)$ is compact iff $\Delta_{A,B}(t)$ is compact for $t \geq 0$.*

We can characterize the norm-continuity in Hilbert space analogously to Theorem 2.5 in [190] and the proof is just a simple modification.

Proposition 10.2.1. *Let A and B generate cosine functions $C(t,A)$ and $C(t,B)$, respectively, on a Hilbert space H, and let $\|C(t,A)\|, \|C(t,B)\| \le Me^{\omega t}$ for some constants $M \ge 1, \omega \in \mathbb{R}$. Then $C(t,A) - C(t,B)$ is norm-continuous for $t > 0$ iff for every $\sigma > \omega$,*

$$\lim_{|r|\to\infty} \|(\sigma + ir)(R((\sigma+ir)^2, A) - R((\sigma+ir)^2, B))\| = 0$$

and

$$\lim_{n\to\infty} \int_n^\infty \|(\sigma \pm ir)(R((\sigma\pm ir)^2, A) - R((\sigma\pm ir)^2, B))x\|^2 dr = 0,$$
$$\lim_{n\to\infty} \int_n^\infty \|(\sigma \pm ir)(R((\sigma\pm ir)^2, A^*) - R((\sigma\pm ir)^2, B^*))y\|^2 dr = 0$$

uniformly for $x \in H, y \in H^*$ with $\|x\|, \|y\| \le 1$.

Theorem 10.2.2 ([191])**.** *Let $S(t,A)$ and $S(t,B)$ be the sine functions of $C(t,A)$ and $C(t,B)$, respectively. Then $S(t,A) - S(t,B)$ is compact for $t > 0$ iff $R(\lambda;A) - R(\lambda;B)$ is compact for $\lambda > w^2$.*

Now we prove a similar result for the cosine as in [190, Proposition 2.7].

Proposition 10.2.2 ([191])**.** *Suppose that $\Delta_{A,B}(t)$ is compact for $t > 0$ and norm-continuous at $t = 0$. Then*

$$\lim_{h\to 0} \|\Delta_{A,B}(t+h) - 2\Delta_{A,B}(t) + \Delta_{A,B}(t-h)\| = 0 \text{ for any } t \ge 0. \tag{10.5}$$

Proof. We have

$$\begin{aligned}
& \Delta_{A,B}(t+h) + \Delta_{A,B}(t-h) - 2\Delta_{A,B}(t) \\
= \; & \Big(C(t+h,A) + C(t-h,A)\Big) - \Big(C(t+h,B) + C(t-h,B)\Big) - 2\Delta_{A,B}(t) \\
= \; & 2C(t,A)C(h,A) - 2C(t,B)C(h,B) - 2\Delta_{A,B}(t) \\
= \; & 2\Big(C(t,A)C(h,A) - C(h,A)C(t,B)\Big) + 2\Big(C(h,A)C(t,B) - C(t,B)C(h,B)\Big) - 2\Delta_{A,B}(t) \\
= \; & 2C(h,A)\Delta_{A,B}(t) + 2\Delta_{A,B}(h)C(t,B) - 2\Delta_{A,B}(t) \\
= \; & 2[C(h,A) - I]\Delta_{A,B}(t) + 2\Delta_{A,B}(h)C(t,B) \to 0 \qquad \text{as } h \to 0.
\end{aligned}$$

Remark 10.2.1. In the proof of Theorem 10.2.1, we actually used (10.5), but not the norm-continuity of $\Delta_{A,B}(\cdot)$.

Theorem 10.2.3 ([191])**.** *Let $\Delta_{A,B}(t)$ be norm-continuous in t at 0. Then $\Delta_{A,B}(t)$ is compact for $t > 0$ iff $R(\lambda;A) - R(\lambda;B)$ is compact for $\lambda > w^2$ and (10.5) holds.*

Proposition 10.2.3 ([191]). *Suppose that the assumptions of Theorem 12.2.1 (resp. Theorem 12.3.1) below hold. Then $\Delta_{A,A(I+B)}(t)$ (resp. $\Delta_{A,(I+B)A}(t)$) is compact for $t > 0$ iff $\Delta_{A,A(I+B)}(t)$ (resp. $\Delta_{A,(I+B)A}(t)$) satisfies (10.5) and $R(\lambda; A) - R(\lambda; A(I+B))$ (resp. $R(\lambda; A) - R(\lambda; (I+B)A)$) is compact for λ large enough.*

We complete this section by comparing the results on the difference of semigroups and cosine operator functions. We first consider bounded perturbations. It is well known that if A generates a C_0-semigroup $\exp(tA)$, then $A+B$, $B \in B(E)$, also generates a C_0-semigroup $\exp(t(A+B))$. It was shown in [302] that $\exp(t(A+B)) - \exp(tA)$ is norm-continuous for $t > 0$ if it is compact for $t > 0$. As for the cosine case, the compactness hypothesis can be removed.

Theorem 10.2.4 ([191]). *Let A be a generator of a cosine function $C(t, A)$, and let $B \in B(E)$. Then $\Delta_{A+B,A}(t)$ is norm-continuous in $t \in \mathbb{R}$.*

Combining Theorems 10.2.1 and 10.2.4, we have the following.

Theorem 10.2.5 ([191]). *Let $B \in B(E)$, and let A generate a cosine function. Then $\Delta_{A+B,A}(t)$ is compact for $t > 0$ iff $R(\lambda; A+B) - R(\lambda; A)$ is compact for λ large enough.*

Proposition 10.2.4 ([191]). *Suppose that C_0-semigroups $\exp(tA)$ and $\exp(tB)$ commute and $D(B) \subseteq D(A)$. Assume that $\exp(tB)$ is a C_0-group. If $\Theta(t) := \exp(tA) - \exp(tB)$ is compact for all $t > 0$, then $A = B + K$, where the operator K is compact.*

Proof. One can write $\exp(-tB)\Theta(t) = \exp(-tB)\exp(tA) - I$, $t \in \mathbb{R}_+$. By the assumption, the operator $\exp(-tB)\exp(tA) - I$ is compact for any $t > 0$, and, moreover, $\exp(-tB)\exp(tA)$ is a C_0-semigroup with the generator $A - B$. It follows from [111] that the operator $A - B$ is compact.

For cosines with bounded generators, we have the following characterization.

Proposition 10.2.5 ([191]). *Let $B \in B(E)$. Then $\Delta_{A,B}(t)$ is compact iff $A - B$ is compact.*

Proof. The compactness of $\Delta_{A,B}(t)$ for any $t > 0$ implies (see [284]) that $R(\mu, A) - R(\mu, B)$ is compact for some μ. In such a case, the operator $I - (\mu I - B)R(\mu, A)$ is compact. This means that $(\mu I - B)R(\mu, A)$ is a Fredholm operator of index 0, i.e., it has a closed range $\mathcal{R}((\mu I - B)R(\mu, A)) = E$. Since $\mu I - B$ is one-to-one on E, we obtain $\mathcal{R}(R(\mu, A)) = E$. By the Banach theorem, $\mu I - A$ is bounded. Since the operator A is bounded, we have $\left\| \frac{2}{t^2} \int_0^t S(s, A)\, ds - I \right\| \to 0$ as $t \to 0$. Hence the operator $\int_0^t S(s, A)\, ds$ is invertible. If $\Delta_{A,B}(t)$ is compact, then $B\int\limits_0^t (S(s,B)-S(s,A))\,ds$ is also compact. Now from

$$\Delta_{A,B}(t) = (A-B)\int_0^t S(s,A)\,ds - B\int_0^t (S(s,B)-S(s,A))\,ds$$

it follows that the difference $A-B$ is a compact operator.

Conversely, if $A-B$ is a compact operator, then A is bounded and

$$R(\lambda,A) - R(\lambda;B) = R(\lambda;A)(B-A)R(\lambda,B)$$

is compact; therefore, the compactness of $\Delta_{A,B}(t)$ follows from Theorem 10.2.1.

In the following example both A and B generate C_0-semigroups and C_0-cosine functions. The operator $\exp(tA)-\exp(tB)$ is compact for all $t>0$, but $C(t,A)-C(t,B)$ is not compact.

Example 10.2.1 ([191]). Let $E=l^1$ and $\{e_n\}$ be the standard basis for it, i.e. $e_n=(0,...,0,1,0,...,0,...)$, where 1 is in the nth coordinate. Let

$$Ax := \sum_{n=1}^{\infty} -n(x,e_n)e_n, \quad Bx := \sum_{n=1}^{\infty} -(n+n^2)(x,e_n)e_n,$$

where $x=(x^1,x^2,...,x^n,...)$ and $(x,e_n)=x^n$, $\|x\|_{l^1}=\sum\limits_{i=1}^{\infty}|x^i|$. Then the C_0-semigroups generated by them are

$$\exp(tA)x = \sum_{n=1}^{\infty} e^{-nt}(x,e_n)e_n, \quad \exp(tB)x = \sum_{n=1}^{\infty} e^{-(n+n^2)t}(x,e_n)e_n$$

and the C_0-cosine functions are given by the formulas

$$C(t,A)x = \sum_{n=1}^{\infty} \cos(nt)(x,e_n)e_n, \quad C(t,B)x = \sum_{n=1}^{\infty} \cos(n+n^2)t(x,e_n)e_n.$$

Since $e^{-nt}-e^{-(n+n^2)t}\to 0$ as $n\to\infty$, the operator $\exp(tA)-\exp(tB)$ can be approximated in norm by a sequence of operators $S_N(t)x=\sum\limits_{n=1}^{N}(e^{-nt}-e^{-(n+n^2)t})(x,e_n)e_n$ with ranges of finite dimension. Therefore, the operator $\exp(tA)-\exp(tB)$ is compact for $t\geq 0$. However, $C(t,A)-C(t,B)$ is not compact. Indeed, take $t=\pi/2$ and choose $\{y_k\}:=\{(\delta_n^{2k+1})\}\in l^1$, where δ_i^j is the Kronecker delta. Now, for $n=2k+1$, we obtain $nt=k\pi+\pi/2$ and $(n+n^2)t=2k^2\pi+3k\pi+\pi$. Therefore, we have $\cos(nt)-\cos((n+n^2)t)=\pm 1$; if k can be divided exactly by 2, then we choose $+$; otherwise, we choose $-$. Thus,

$$(C(\pi/2,A)-C(\pi/2,B))y_k = \{[\cos(nt)-\cos(n+n^2)t]\delta_n^{2k+1}\} = \{\pm\delta_n^{2k+1}\},$$

which means $\|[C(\pi/2,A)-C(\pi/2,B)](y_k-y_m)\|_{l^1}=2$ for $k\neq m$. Therefore, we cannot choose a convergent subsequence from the sequence $\{[C(\pi/2,A)-C(\pi/2,B)]y_k\}$. Also, we can see that $C(t,A)-C(t,B)$ is not norm-continuous in t. Indeed, for each $t > 0$, define $s_k = t + \dfrac{1}{k+k^2}$. Then $s_k \to t$ as $k \to \infty$ and

$$\begin{aligned}
& \Big\| [C(t,A) - C(t,B)] - [C(s_k,A) - C(s_k,B)] \Big\|_{B(l^1)} \\
= & \Big\| \{[\cos(nt) - \cos(ns_k)] - [\cos(n+n^2)t - \cos(n+n^2)s_k]\}_{n=1}^{\infty} \Big\|_{l^\infty} \\
\geq & \ 2\Big| \sin\left(\frac{k}{2}(s_k+t)\right)\sin\left(\frac{k}{2}(s_k-t)\right) - \sin\left(\frac{k+k^2}{2}(s_k+t)\right)\sin\left(\frac{k+k^2}{2}(s_k-t)\right) \Big| \\
= & \ 2\Big| \sin\left(\frac{k}{2}(s_k+t)\right)\sin\left(\frac{1}{2(1+k)}\right) - \sin((k+k^2)t+1/2)\sin\frac{1}{2} \Big|.
\end{aligned}$$

It is clear that $\sin\dfrac{1}{2(1+k)} \to 0$ as $k \to \infty$. But $\sin((k+k^2)t+1/2)$ does not converge to 0 as $k \to \infty$ for every $t > 0$! To prove this, we suppose the contrary: $\sin((k+k^2)t+1/2) \to 0$ as $k \to \infty$. Then $\sin((k+1+(k+1)^2)t+1/2) \to 0$ as $k \to \infty$. Now, since

$$\begin{aligned}
& \sin((k+1+(k+1)^2)t+1/2) = \sin((k+k^2)t+1/2+2(k+1)t) \\
= & \ \sin((k+k^2)t+1/2))\cos(2(k+1)t) + \cos((k+k^2)t+1/2))\sin(2(k+1)t),
\end{aligned}$$

we obtain $\sin(2(k+1)t) \to 0$ as $k \to \infty$, since $\cos((k+k^2)t+1/2)$ cannot converge to 0 according to the relation $\sin^2 x + \cos^2 x = 1$. Hence $\sin(2(k+1+1)t) \to 0$ as $k \to \infty$. Thus, from $\sin(2(k+1+1)t) = \sin(2(k+1)t)\cos(2t) + \cos(2(k+1)t)\sin(2t)$ we obtain $\sin(2t) \to 0$ as $k \to \infty$. Therefore, we have $t = n\pi/2$ for some $n \in \mathbb{N}$. But for such t, we find that $\sin((k+k^2)t+1/2)) = \pm\sin(1/2)$, which contradicts our assumption of the convergence to 0. This means that $C(t,A) - C(t,B)$ is not norm-continuous.

The converse does not hold: we have the following proposition.

Proposition 10.2.6 ([191]). *Suppose that A and B generate cosine functions $C(t,A)$ and $C(t,B)$. If $C(t,A) - C(t,B)$ is compact for $t > 0$, then $\exp(tA) - \exp(tB)$ is compact. Moreover, if $C(t,A) - C(t,B)$ is norm-continuous for $t > 0$, then $C(t,A) - C(t,B)$ is compact for $t > 0$ iff $\exp(tA) - \exp(tB)$ is compact.*

Proof. Since $C(t,A) - C(t,B)$ is compact, it follows from [284] that $S(t,A) - S(t,B) = \int_0^t (C(s,A) - C(s,B))ds$ is also compact, which implies that $R(\lambda;A) - R(\lambda;B)$ is compact by Theorem 10.2.2. Moreover, since both $\exp(tA)$ and $\exp(tB)$ are analytic, $\exp(tA) - \exp(tB)$ is norm-continuous, and the compactness of $\exp(tA) - \exp(tB)$ follows from Theorem 2.3 of [190]. If, in addition, $C(t,A) - C(t,B)$ is norm-continuous, then the compactness of $\exp(tA) - \exp(tB)$ implies that $R(\lambda;A) - R(\lambda;B)$ is compact. Now the compactness of $C(t,A) - C(t,B)$ follows from Theorem 10.2.1.

The following proposition extends Proposition 2.7 from [190].

Proposition 10.2.7 ([191]). *Suppose that $D(A) \subseteq D(B)$, where A generates an analytic C_0-semigroup* $\exp(tA)$ *and B is a generator of a C_0-semigroup* $\exp(tB)$. *If $\Theta(t) := \exp(tA) - \exp(tB)$ is compact for $t > 0$, then $\Theta(t)$ is norm-continuous for $t \geq 0$.*

10.3. Compactness of the Families $F(\cdot)$ and $G(\cdot)$

Definition 10.3.1. A C_0-family of multiplicative perturbations $F(\cdot)$ (resp. a C_0-family of additive perturbations $G(\cdot)$) is said to be *compact* if the operators $F(t)$ (resp. $G(t)$) are compact for each $t \in \overline{\mathbb{R}}_+$.

Proposition 10.3.1. *If a C_0-sine operator function $S(\cdot, A)$ is compact and if a C_0-family of multiplicative perturbations $F(\cdot)$ is norm-continuous at zero, then $F(\cdot)$ is compact. The same is true for a C_0-family of additive perturbations.*

Proof. Integrating relation (2.1) in t from 0 up to τ, we obtain

$$\int_{\tau}^{\tau+h} F(\eta)x\,d\eta - \int_{\tau-h}^{\tau} F(\eta)x\,d\eta = 2S(\tau, A)F(h)x. \tag{10.6}$$

The compactness of $S(\cdot, A)$ implies the compactness of the left-hand side of (10.6) for any $\tau, h \in \overline{\mathbb{R}}_+$. Since $F(\cdot)$ is uniformly continuous, we can take in (10.6) the derivative in τ without loss of the compactness property, since the obtained left-hand side of (10.6) remains a compact operator. Using the condition $F(0) = 0$ and the uniform continuity of $F(\cdot)$ and tending τ to zero, we obtain that $F(h)$ is compact for each $h \in \overline{\mathbb{R}}_+$.

The requirement of the uniform continuity in Proposition 10.3.1 is also necessary, as the following proposition shows.

Proposition 10.3.2 ([239]). *If a family of multiplicative perturbations $F(\cdot)$ is compact, then the family $F(\cdot)$ is uniformly continuous on $\overline{\mathbb{R}}_+$.*

Proof. Since $F(\cdot)$ is compact, the Laplace transform $\hat{F}(\cdot)$ is also compact (see [284]). By formula (iv) of Proposition 2.4.1, the assertion is proved, since the strong convergence becomes uniform after postmultiplying by a compact operator.

Proposition 10.3.3 ([239]). *Let a C_0-cosine operator function $C(\cdot, A)$ on a Banach space E be such that each of the families of multiplicative perturbations $F(\cdot)$ (or the C_0-families of additive perturbations $G(\cdot)$) for $C(\cdot, A)$ is compact. Then E is finite-dimensional.*

Proof. By assumption, two particular families of multiplicative perturbations

$$F_1(t) = C(t, A) - I \quad \text{and} \quad F_2(t) = \int_0^t S(s, A)\,ds$$

are compact. Then since we know (see Theorem 10.1.2) that the operator $C(t,A)-I$ is compact for all $t \in \mathbb{R}_+$ iff the generator A is compact, the family $C(\cdot,A)$ is norm-continuous on $\overline{\mathbb{R}}_+$. Therefore, the operator $C(0,A)=I$, being the limit in norm of compact operators $2t^{-2}F_2(t)$ as $t \to 0$ is compact. This implies that E is finite-dimensional.

Chapter 11

ADJOINT COSINE OPERATOR FUNCTIONS

Cosine operator functions adjoint in the sense of Phillips were little considered in the literature: on one hand, because of close analogies with the theory of C_0-semigroups of operators, and on the other hand, because of the absence of very valuable applications. But we essentially use the properties of $C(\cdot,A)^{\odot}$ in the perturbation theory when considering the lifting theorems.

11.1. C_0-Cosine Operator Functions Adjoint in the Sense of Phillips

Denote by $C(t,A)^{\odot}$, $t \in \mathbb{R}$, the restriction $C(t,A)^*|_{E^{\odot}}$, $t \in \mathbb{R}$, where $E^{\odot} \subseteq E$ is the subspace on which the adjoint family $C(\cdot,A)^*$ is strongly continuous at zero.

Proposition 11.1.1 ([222]). *For a C_0-cosine operator function $C(\cdot,A)$ given on E, we have*:

(i) *if $x^* \in D(A^*)$, then for any $t \in \mathbb{R}$,*

$$C(t,A)^*x^* \in D(A^*) \quad \textit{and} \quad A^*C(t,A)^*x^* = C(t,A)^*A^*x^*,$$

and the following relation holds for any $x \in E$:

$$\langle x, (C(t,A)^* - I^*)x^* \rangle = \int_0^t (t-s)\langle x, C(s,A)^*A^*x^*\rangle ds;$$

(ii) *the inclusion $x^* \in D(A^*)$ holds iff there exists the limit*

$$w^*\text{-}\lim_{s\to 0+} (2/s^2)(C(s,A)^* - I^*)x^* = y^*, \quad \textit{and, moreover,} \quad A^*x^* = y^*.$$

Theorem 11.1.1 ([222]). *In the notation of Proposition* 11.1.1, *we have the following*:

(i) *the subspace $E^{\odot} = \overline{D(A^*)}$, where the closure is understood in the strong topology of the space E^**;

(ii) *the subspace $E^{\odot}$ is invariant with respect to $C(t,A)^*$, and $C(t,A)^{\odot}$, $t \in \mathbb{R}$, is a C_0-cosine operator function on $E^{\odot}$*;

(iii) *the generator $A^{\odot}$ of the C_0-cosine operator function $C(\cdot,A)^{\odot}$ is maximal among the restrictions of the operator A^* to $E^{\odot}$ (i.e., $A^{\odot}$ is a part of the operator A^* on $E^{\odot}$)*;

(iv) *if E is reflexive, then $E^{\odot} = E^*$ and $A^{\odot} = A^*$*;

(v) *for each $t \in \mathbb{R}_+$, the operator $C(t,A)^*$ is the w^*-closure of the operator $C(t,A)^{\odot}$.*

Proposition 11.1.2 ([222])**.** *For any $x^* \in D(A^*)$, we have*

$$\|(C(t,A)^* - I^*)x^*\| \le (t^2/2))\|A^*x^*\| \cdot \sup_{0\le s\le t} \|C(s,A)\|.$$

11.2. Adjoint Families

Definition 11.2.1. An operator function $K(t)$, $t \in \mathbb{R}$, given on the space E^* and satisfying the conditions $K(0) = I^*$ on it and the functional cosine equation (see (i) on p. 83), is said to be a *w^*-continuous C_0-cosine operator function* if for each $t \in \mathbb{R}_+$, the operator $K(t)$ is continuous on the space E^* in the w^*-topology (we write w^*-w^*-continuous), and for any $x^* \in E^*$, the function $t \to K(t)x^*$ is w^*-continuous on E^* in $t \in \mathbb{R}$.

Proposition 11.2.1 ([259])**.** *An operator $Q \in L(E^*)$ is w^*-w^*-closed iff it is adjoint to a densely defined closed operator $\mathcal{Q} \in \mathcal{C}(E)$. Moreover, $D(Q) = E^*$ iff $D(\mathcal{Q}) = E$, and in this case, both $\mathcal{Q}$ and Q are bounded, and, moreover, $\|\mathcal{Q}\| = \|Q\|$.*

Theorem 11.2.1 ([259])**.** *An operator function $K(t)$ on E^* is a w^*-continuous C_0-cosine operator function iff $K(\cdot) = C(\cdot, A)^*$, where $C(\cdot, A)$ is a certain C_0-cosine operator function. If Q is a generator of $K(t)$, $t \in \mathbb{R}$ (in the sense of the w^*-topology) and A is a generator of a C_0-cosine operator function $C(\cdot, A)$, then $Q = A^*$.*

For a w^*-continuous C_0-cosine operator function with a generator Q, we accept the notation $K(t, Q)$.

Proposition 11.2.2 ([259])**.** *In the notation of Theorem* 11.2.1, *the following conditions are equivalent*:

(i) *an element $x^* \in D(Q)$;*

(ii) $\|K(t,Q)x^* - x^*\| = O(t^2)$ *as* $t \to 0$;

(iii) $\overline{\lim}_{t\to 0+} t^{-2}\|K(t,Q)x^* - x^*\| < \infty$.

Proposition 11.2.3 ([259])**.** *Let $Q \in L(E^*)$. An operator Q generates a w^*-continuous C_0-cosine operator function iff it is w^*-densely defined, w^*-w^*-closed, and there exist constants $M > 0$ and $\omega > 0$ such that for $\operatorname{Re}\lambda > \omega$, the point $\lambda^2 \in \rho(Q)$ and*

$$\left\| \frac{d^m}{d\lambda^m}(\lambda(\lambda^2 I^* - Q)^{-1}) \right\| \le \frac{Mm!}{(\lambda-\omega)^{m+1}}, \quad m \in \mathbb{N}.$$

Proposition 11.2.4 ([259])**.** *If a set $\mathcal{D} \subseteq D(Q)$ is w^*-dense in $D(Q)$ and is invariant with respect to a w^*-continuous C_0-cosine operator function $K(\cdot, Q)$, then $\mathcal{D}$ is the w^*-core of the operator Q.*

Proposition 11.2.5 ([259])**.** *The following conditions are equivalent for w^*-continuous C_0-cosine operator functions $K(t, Q_1)$ and $K(t, Q_2)$:*

(i) *the domains* $D(Q_1) \subseteq D(Q_2)$;

(ii) $\|(K(t,Q_1) - K(t,Q_2))x^*\| = O(t^2)$ *as* $t \to 0$ *for any* $x^* \in D(Q_1)$.

Proposition 11.2.6 ([259]). *The following conditions are equivalent:*

(i) $\|K(t,Q_1) - K(t,Q_2)\| = O(t^2)$ *as* $t \to 0$;

(ii) $D(Q_1) = D(Q_2)$ *and* $Q_2 - Q_1$ *is a bounded operator on* $D(Q_1)$;

(iii) $D(Q_1) \subseteq D(Q_2)$ *and* $Q_2 - Q_1$ *is a bounded operator on* $D(Q_1)$;

(iv) $D(Q_2) \subseteq D(Q_1)$ *and* $Q_2 - Q_1$ *is a bounded operator on* $D(Q_2)$.

Moreover, in these cases,

$$\|Q_2 - Q_1\| \le \varliminf_{t\to 0} 2t^{-2}\|K(t,Q_1) - K(t,Q_2)\| \le \sup_{t\in\mathbb{R}_+} 2t^{-2}\|K(t,Q_1) - K(t,Q_2)\|,$$

and, moreover, the equalities are attained, e.g., at contractive w^**-continuous* C_0*-cosine operator functions.*

Proposition 11.2.7 ([259]). *If* $K(t,Q_1) - K(t,Q_2) = o(t^2)$ *as* $t \to 0$, *then*

$$K(t,Q_1) = K(t,Q_2), \ t \in \mathbb{R}.$$

Proposition 11.2.8 ([259]). *For a* w^**-continuous* C_0*-cosine operator function* $K(t,Q)$, $t \in \mathbb{R}$, *we have*

$$\mathcal{N}(Q) = \bigcap_{t>0} \mathcal{N}\Big(K(t,Q) - I^*\Big),$$

and $w^*\text{-}\mathrm{cl}(\mathcal{R}(Q))$ *is* $w^*\text{-}\mathrm{cl}\left(\bigcup_{t>0}\mathcal{R}(K(t,Q) - I^*)\right)$.

If E *is a Grothendieck space, then*

$$\overline{\mathcal{R}(Q)} = w\text{-}\mathrm{cl}\left(\bigcup_{t>0}\mathcal{R}\Big(K(t,Q) - I^*\Big)\right) = \overline{\mathrm{span}\{\mathcal{R}(K(t,Q) - I^*) : t \in \mathbb{R}_+\}},$$

where the closure is understood in the strong topology of E^*.

Introduce the following notation:

$$Q_s^1 := s\text{-}\lim_{t\to\infty} t^{-1}S(t,A)^*, \quad Q_w^1 := w\text{-}\lim_{t\to\infty} t^{-1}S(t,A)^*, \quad Q_{w^*}^1 := w^*\text{-}\lim_{t\to\infty} t^{-1}S(t,A)^*.$$

Proposition 11.2.9 ([259]). *Assume that for* $Q = A^*$, *the following conditions hold:*

(a) $\|S(t,A)\| = O(t) \quad$ *as* $\quad t \to \infty$;

(b) $w^*\text{-}\lim_{t\to\infty} t^{-1}((K(t+s,Q) - K(t-s,Q))S(s,A)^*x^* = 0$ *for all* $x^* \in E$ *and* $s \in \mathbb{R}_+$.

Then:

(i) $Q_s^1 \subseteq Q_w^1 \subseteq Q_{w^*}^1$ *are projections, and, moreover,*

$$\|Q_{w^*}^1\| \le \varliminf_{t\to\infty} t^{-1}\|S(t,A)\|, \quad \text{and} \quad D(Q_s^1) \subseteq D(Q_w^1) \quad \text{and} \quad D(Q_{w^*}^1)$$

are strongly closed;

(ii) $\mathcal{R}(Q_s^1) = \mathcal{R}(Q_w^1) = \mathcal{R}(Q_w^1) = \mathcal{N}(Q)$, $\mathcal{N}(Q_s^1) \subseteq \mathcal{N}(Q_w^1) \subseteq \overline{\mathcal{R}(Q)}$, *and*

$$\mathrm{Sp} := \overline{\mathrm{span}\{\mathcal{R}(K(t,Q) - I^*) : t > 0\}} \subseteq \mathcal{N}(Q_{w^*}^1) \subseteq w^*\text{-}\mathrm{cl}(\mathcal{R}(Q)).$$

If we replace condition (b) by a stronger condition (b′) $s\text{-}\lim_{t\to\infty} \frac{1}{t}(K(t+s,Q) - K(t-s,Q))S(s,A)^* = 0$ *for all* $s \in \mathbb{R}_+$, *then* $\mathrm{Sp} \subseteq \mathcal{N}(Q_s^1)$.

Proposition 11.2.10 ([259])**.** *Let conditions (a) and (b) of Proposition 11.2.9 hold, and let E be a Grothendieck space. Then*

$$\mathcal{R}(Q_{w^*}^1) = \mathcal{N}(Q), \quad \mathcal{N}(Q_{w^*}^1) = \overline{\mathcal{R}(Q)} \quad \textit{and} \quad D(Q_{w^*}^1) = \mathcal{N}(Q) \oplus \overline{\mathcal{R}(Q)} = E^*.$$

If, in addition, the condition (b′) holds, then $K(\cdot,Q)$ is strongly $(C,1)$ ergodic, i.e., $D(Q_s^1) = E^*$.

Definition 11.2.2. Denote by Q_s^2, Q_w^2, and $Q_{w^*}^2$ the Cesaro $(C,2)$-averagings of the Cesaro $(C,1)$-averagings Q_s^1, Q_w^1, and $Q_{w^*}^1$ defined in the corresponding way.

Proposition 11.2.11 ([259])**.** *Let $\|T(t,A)\| = O(t^2)$ as $t \to \infty$, and let $s\text{-}\lim_{t\to\infty} t^{-2}K(t,Q)x^* = 0$ for all $x^* \in D(Q)$. Then $Q_s^2 = Q_w^2 \subseteq Q_{w^*}^2$ are bounded projections such that*

$$\|Q_{w^*}^2\| \le \varliminf_{t\to\infty} 2t^{-2}\|T(t,A)\|, \quad \mathcal{R}(Q_s^2) = \mathcal{R}(Q_{w^*}^2) = \mathcal{N}(Q),$$
$$\mathcal{R}(Q) = \mathcal{N}(Q_s^2) \subseteq \mathcal{N}(Q_{w^*}^2) \subseteq w^*\text{-}\mathrm{cl}(\mathcal{R}(Q)).$$

The subspaces $D(Q_s^2)$ and $D(Q_{w^*}^2)$ are strongly closed in E^*, and

$$D(Q_s^2) = \mathcal{N}(Q) \oplus \overline{\mathcal{R}(Q)} = \{x^* \in E^* : \exists\, t_n \to \infty : \lim_{n\to\infty} 2t_n^{-2}T(t_n)x^* \textit{ exists}\}.$$

Proposition 11.2.12 ([259])**.** *Let*

$$Gx^* := s\text{-}\lim_{t\to 0+} t^{-1}S(t,A)^*x^*$$

for those $x^* \in E^*$ for which the limit exists. Then for a w^*-continuous C_0-cosine operator function $K(\cdot,Q)$ with $Q = A^*$, we have

$$D(G) = \overline{\bigcup_{t>0} \mathcal{R}(S(t,A)^*)} = \{x^* \in E^* : \exists\, t_n \to \infty : w^*\text{-}\lim_{n\to\infty} t_n^{-1}S(t_n,A)^*x^* \textit{ exists}\}.$$

Moreover, $G = I_{D(G)}$.

PERTURBATIONS OF C_0-COSINE OPERATOR FUNCTIONS

The perturbation theory of C_0-cosine operator functions differs from that of C_0-semigroups in an interesting way. On one hand, the generator of a C_0-cosine operator function lies in a more narrow class of operators than $\mathcal{G}(M,\omega)$; for example, it always generates an analytic C_0-semigroup, and, therefore, its fractional powers $(-A)^\alpha$, $0 \le \alpha \le 1$, are defined in a sufficiently simple way. On the other hand, it is not clear up to now whether the M. Watanabe perturbation is the strongest perturbation or not, and what happens with SV of a family of multiplicative perturbations if $\mathrm{SV}(F(\cdot),t) = O(t^\alpha)$, $t \to 0+$, for a certain $0 < \alpha < 1$.

12.1. General Multiplicative Theorems

We quote the following theorem (see [192, Theorem 3.10]) for $C(\cdot, A)$, which is convenient for practical applications and which will be repeatedly used below.

Theorem 12.1.1. *A C_0-cosine operator function $C(\cdot,A)$ satisfying the estimate $\|C(t,A)\| \le Me^{\omega t}$ for all $t \ge 0$ is a C_0-cosine operator function with a generator A iff the condition $\lambda > \omega$ implies $\lambda^2 \in \rho(A)$ and we have*

$$\lambda(\lambda^2 I - A)^{-1} = \int_0^\infty e^{-\lambda t} C(t,A)\, dt.$$

Proposition 12.1.1 ([240]). *Let A be a densely defined closed linear operator on a Banach space E, and let $\Re \in B(X)$. The following assertions hold*:

(i) *if the operator $\Re A$ generates a cosine operator function $\tilde{C}(\cdot,A)$, then the operator $A\Re$ also generates a C_0-cosine operator function*;

(ii) *if the operator $A\Re$ generates a cosine operator function $\hat{C}(\cdot,A)$, and for a certain real λ, the operator $\lambda - \Re A$ is invertible, then the operator $\Re A$ also generates a C_0-cosine operator function*;

(iii) *if the operator $A\Re$ generates a cosine operator function $\check{C}(\cdot,A)$ and $D((A\Re)^*) = D(A^*)$, then the operator $\Re A$ also generates a C_0-cosine operator function.*

Definition 12.1.1. We say that an operator $\Re \in B(E)$ belongs to the class $M1(A)$ of multiplicative perturbations of the generator A of a C_0-cosine operator function $C(\cdot,A)$ if the operator $B = \Re - I$ satisfies the following Condition $(M1)$: for all continuous functions $f \in C([0,t];E)$

$(M1_a)$ $\int\limits_0^t S(t-s,A)B\,f(s)ds \in D(A)$,

$(M1_b)$ $\left\| A \int\limits_0^t S(t-s,A)Bf(s)ds \right\| \le M\gamma_B(t)\|f\|_{[0,t]}$, where $\gamma_B : [0,\infty) \to [0,\infty)$ is some continuous nondecreasing function with $\gamma_B(0) = 0$ and $\|f\|_{[0,t]} = \sup_{0\le s\le t} \|f(s)\|$.

Remark 12.1.1. If $B+I \in M1(A)$, then $\|C(t+h,A)B - C(t,A)B\| \to 0$ as $h \to 0$ for any t. To see this, we first set $f(t) = x$ for $t \ge 0$ in $(M1_b)$. It follows that $\|(C(h,A) - I)B\| \to 0$ as $h \to 0$. This, together with the fact that $(C(\cdot,A) - I)B$ is a C_0-family of multiplicative perturbations, proves the assertion.

Theorem 12.1.2 ([240])**.** *Let A be the infinitesimal generator of a C_0-cosine operator function $C(\cdot,A)$ on E. If an operator $\Re$ belongs to $M1(A)$, then both $A\Re$ and $\Re A$ are generators of C_0-cosine operator functions. Moreover, the C_0-cosine operator function $\Im(\cdot)$ generated by $A\Re$ satisfies $\|\Im(t) - C(t,A)\| = O(\gamma_B(t))$ $(t \to 0^+)$.*

Remark 12.1.2. (i) If $(M1_a)$ and $(M1_b)$ hold for all functions in a dense subset of $C([0,t];E)$, then because of the closedness of A, we easily see that they actually hold for all f in $C([0,t];E)$. Hence since $(M1_a)$ holds for all f in $C^1([0,t];E)$, which is dense in $C([0,t];E)$, Condition $(M1)$ can be replaced by the equivalent condition:

$$\left\| A \int_0^t S(t-s,A)Bf(s)ds \right\| \le M\gamma_B(t)\|f\|_{[0,t]} \quad \text{for all } f \in C^1([0,t];E). \tag{12.1}$$

Thus, we only need to verify Condition $(M1')$ in practical applications.

(ii) If $(M1)$ holds with some $\gamma_B(t) = o(t^2)$, then $\Im(\cdot) \equiv C(\cdot,A)$ (see [240, Corollary 3.6]), so that $A(I+B) = A$ and AB=0. Conversely, the latter condition implies $C(\cdot,A)B = B$, and hence $A\int_0^t S(t-s,A)Bf(s)ds \equiv 0$ for all $f \in C([0,t],X)$. Thus, $(M1)$ holds with some $\gamma_B(t) = o(t^2)$ iff $AB = 0$, and in this case, $(M1)$ actually holds with $\gamma_B(\cdot) \equiv 0$.

Let $(Z, |\cdot|)$ be a Banach space satisfying Condition (Z) with respect to $C(\cdot,A)$:

(Z_a) Z is continuously embedded in E,

(Z_b) for all continuous functions $\phi \in C([0,t],Z)$,

$$\int_0^t S(t-s,A)\phi(s)ds \in D(A),$$

(Z_c) $\left\| A \int_0^t S(t-s,A)\phi(s)ds \right\| \le \gamma(t) \sup_{0\le s\le t} |\phi(s)|_Z,$

where $\gamma(\cdot) : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function with $\gamma(0) = 0$.

It is easy to verify Condition (Z) for the spaces $\mathcal{D}(A)$ and the Favard class $(\mathrm{Fav}_{C(\cdot,A)}, |\cdot|_{\mathrm{Fav}_{C(\cdot,A)}})$. Indeed, if $Z = \mathcal{D}(A)$, then (Z) holds with $\gamma(t) = O(t^2)$ as $t \to 0^+$.

Corollary 12.1.1. *If Z is a Banach space satisfying Condition (Z), then $I + B(E,Z) \subseteq M1(A)$, so that for every $B \in B(E,Z)$, both $A(I+B)$ and $(I+B)A$ are generators of C_0-cosine operator functions.*

Definition 12.1.2. We say that an operator $\Re \in B(E)$ belongs to the class $M2(A)$ of multiplicative perturbations of the generator A of a C_0-cosine operator function $C(\cdot, A)$ if the operator $B = \Re - I$ satisfies

$$\delta_B(t) := \sup\Big\{\int_0^t \|BS(s,A)Ax\|ds : x \in D(A), \|x\| \leq 1\Big\} \to 0 \text{ as } t \to 0^+. \tag{12.2}$$

Remark 12.1.3. As was shown by Fattorini [131], in the case of $E = L_p$, we have $\|AS(t,A)x\| = O(t^{2\alpha-1})$ as $t \to 0$ for $1/2 \leq \alpha \leq 1$ and $x \in D((A - cI)^\alpha)$. Therefore, $\Re \in M2(A)$, e.g., if $B = (A - cI)^{-\beta}$ for $\beta \geq 1/2$.

Theorem 12.1.3 ([240]). *Let A be the infinitesimal generator of a C_0-cosine operator function $C(\cdot, A)$ on E. If an operator $\Re$ belongs to $M2(A)$, then both $\Re A$ and $A\Re$ are generators of C_0-cosine operator functions. Moreover, the cosine function $C_1(\cdot)$ generated by $\Re A$ satisfies $\|C_1(t) - C(t,A)\| = O(\delta_B(t))$ $(t \to 0^+)$.*

12.2. Perturbations by the Family $F(\cdot)$

For any fixed λ and an operator $B \in B(E)$, let $F_{B,\lambda}(\cdot)$ and $G_{B,\lambda}(\cdot)$ be the functions defined by

$$F_{B,\lambda}(t)x := (\lambda^2 I - A)\int_0^t S(s,A)Bx\,ds = \lambda^2 \int_0^t S(s,A)Bxds - (C(t,A) - I)Bx, \;\; x \in E, \, t \geq 0, \tag{12.3}$$

$$G_{B,\lambda}(t)x := B(\lambda^2 I - A)\int_0^t S(s,A)x\,ds = \lambda^2 B\int_0^t S(s,A)xds - B(C(t,A) - I)x, \;\; x \in E, \, t \geq 0. \tag{12.4}$$

Definition 12.2.1. Operator function $f(\cdot)$ is called a function with a *locally bounded semivariation* if for some $t > 0$,

$$\mathrm{SV}(f(\cdot), t) := \sup\left\{\left\|\sum_{j=1}^n [f(t_j) - f(t_{j-1})]x_j\right\| : x_j \in E, \|x_j\| \leq 1\right\} < \infty,$$

where the supremum is taken over all partitions of the interval $[0,t]$. Operator function $f(\cdot)$ is called a function with a *locally bounded strong variation* if for some $t > 0$ and all $x \in E$,

$$\mathrm{Var}(f(\cdot)x, t) := \sup\left\{\sum_{j=1}^n \left\|(f(t_j) - f(t_{j-1}))x\right\| : 0 = t_0 < t_1 < \cdots < t_n = t, n \geq 1\right\} < \infty.$$

Finally, operator function $f(\cdot)$ is called a function with a *locally bounded uniform variation* if for some $t > 0$,

$$\mathrm{Var}(f(\cdot), t) := \sup\left\{\sum_{j=1}^n \left\|(f(t_j) - f(t_{j-1}))\right\| : 0 = t_0 < t_1 < \cdots < t_n = t, n \geq 1\right\} < \infty.$$

The next theorem gives a characterization of $M1(A)$ in terms of the semivariation of $F_{B,\lambda}(\cdot)$.

Theorem 12.2.1 ([240]). *An operator $\Re \in B(E)$ belongs to $M1(A)$, i.e., $B = \Re - I$ satisfies Condition $(M1)$, iff* $\mathrm{SV}(F_{B,\lambda}(\cdot), t) = o(1)$ $(t \to 0^+)$ *for some (and all) $\lambda > \omega$. Moreover, in Condition $(M1_b)$, one can choose $\gamma_B(t) = \mathrm{SV}(F_{B,\lambda}(\cdot), t)$ in the case $\mathrm{SV}(F_{B,\lambda}(\cdot), t) = O(t^2)$, and $\gamma_B(t) = O(t^2)$ in the case $SV(F_{B,\lambda}(\cdot), t) = o(t^2)$.*

Now, from Theorem 12.2.1, we can deduce the following theorem on an additive perturbation.

Theorem 12.2.2 ([240]). *Let A be the generator of a C_0-cosine operator function $C(\cdot, A)$ on E. If $P \in B(\mathcal{D}(A), E)$ is such that*

$$\int_0^t S(t-s, A)Pg(s)ds \in D(A), \tag{12.5}$$

$$\left\| A \int_0^t S(t-s, A)Pg(s)ds \right\| \leq \gamma_P(t) \sup_{0 \leq s \leq t} \|g(s)\|_{\mathcal{D}(A)} \tag{12.6}$$

for all $g \in C([0,t], \mathcal{D}(A))$ and for some function $\gamma_P(\cdot)$ with $\gamma_P(t) = o(1)\,(t \to 0^+)$, then the operators $A + P$ and $A + (A - \lambda I)P(A - \lambda I)^{-1}\,(\lambda > \omega)$ are generators of C_0-cosine operator functions.

Proof. Without loss of generality, we may assume that A is invertible, so that $A + P = (I + PA^{-1})A$. In view of Theorem 12.2.1, we only have to verify Condition $(M1)$ for the operator $B = PA^{-1}$. Indeed, if $f \in C([0,t]; E)$, then $A^{-1}f \in C([0,t], \mathcal{D}(A))$, so that, setting $g = A^{-1}f$ in (12.5) and (12.6), we have

$$\int_0^t S(t-s, A)Bf(s)ds = \int_0^t S(t-s, A)P(A^{-1}f)(s)ds \in D(A)$$

and

$$\left\| A \int_0^t S(t-s, A)Bf(s)ds \right\| \leq \gamma_P(t) \sup_{0 \leq s \leq t} \|A^{-1}f(s)\|_{\mathcal{D}(A)} \leq \gamma_P(t)(\|A^{-1}\| + 1)\|f\|_{[0,t]}.$$

Corollary 12.2.1. *Let A be the generator of a C_0-cosine operator function $C(\cdot)$ on E. If P is a continuous operator acting from $D(A)$ to Z (where Z is a Banach space satisfying condition (Z)), then $A + P$ and $A + (A - \lambda)P(A - \lambda)^{-1}$ $(\lambda > \omega)$ are generators of C_0-cosine operator functions.*

Proof. Let $P \in B(\mathcal{D}(A), Z)$, and let $g \in C([0,t], D(A))$. Then we have $Pg \in C([0,t]; Z)$, and by Condition (Z), $\int_0^t S(t-s, A)Pg(s)ds \in D(A)$ and

$$\left\| A \int_0^t S(t-s)Pg(s)ds \right\| \leq \gamma_P(t) \sup_{0 \leq s \leq t} |Pg(s)|_Z \leq \gamma_P(t)\|P\|_{B(\mathcal{D}(A),Z)} \sup_{0 \leq s \leq t} \|g(s)\|_{\mathcal{D}(A)}.$$

The conclusion now follows from Theorem 12.2.2.

The following theorem gives a characterization of $M2(A)$ in terms of the strong variation of $G_{B,\lambda}(\cdot)$ for $x \in D(A)$; more precisely,

$$\beta(G_{B,\lambda}(\cdot), t) := \sup\{\mathrm{Var}(G_{B,\lambda}(\cdot)x, t);\ x \in D(A), \|x\| \le 1\}. \tag{12.7}$$

Theorem 12.3.1 ([240]). *An operator $\Re \in B(E)$ belongs to $M2(A)$, i.e., $B = \Re - I$ satisfies Condition $(M2)$, iff $\beta(G_{B,\lambda}(\cdot), t) = o(1)$ $(t \to 0^+)$ for some (and all) $\lambda > \omega$. Moreover, $\beta(G_{B,\lambda}(\cdot), t)$ and the function $\delta_B(t)$ in Condition $(M2)$ has the same order of convergence at zero whenever the order does not exceed $O(t^2)$.*

Proof. From (12.4), we see that

$$\mathrm{Var}(G_{B,\lambda}(\cdot)x, t) = \mathrm{Var}(BC(\cdot, A)x, t) + \lambda^2\|B\| \int_0^t \|S(s, A)x\| ds$$

if the variation exists. Since for $x \in D(A)$,

$$\mathrm{Var}(BC(\cdot, A)x, t) = \int_0^t \left\| \frac{d}{ds} BC(s, A)x \right\| ds = \int_0^t \|BS(s, A)Ax\| ds,$$

we have

$$\|\beta(G_{B,\lambda}(\cdot), t) - \delta_B(t)\| \le \lambda^2\|B\| \int_0^t \|S(s, A)\| ds \le \lambda^2\|B\|Me^{\omega t}t^2.$$

Hence $\delta_B(t)$ tends to 0 as $t \to 0^+$ iff $\beta(G_{B,\lambda}(\cdot), t)$ does. They have the same order of convergence at zero whenever one of them has the order less than or equal to $O(t^2)$.

In general, $M1(A)$ and $M2(A)$ are proper subsets of $I + B(E)$. Each of the conditions $M1(A) = I + B(E)$ and $M2(A) = I + B(E)$ is equivalent to the condition that A is bounded. Indeed, if for every $B \in B(E)$, the operator $(I + B)A$ generates a cosine operator function, then for $B = -2I$, we have that $-A$ also generates a C_0-cosine operator function. Hence both A and $-A$ generate analytic C_0-semigroups and, consequently, A is bounded.

From Theorem 12.3.1, we deduce the following additive perturbation theorem.

Theorem 12.3.2 ([240]). *Let A be the generator of a cosine operator function $C(\cdot)$ on E. If P is an operator satisfying the following conditions:*

$$D(A) \subset D(P) \text{ and } P(\lambda^2 I - A)^{-1} \in B(E) \text{ for some } \lambda > \omega; \tag{12.8}$$

$$\theta_P(t) := \sup\left\{ \int_0^t \|PS(s)x\| ds; x \in D(A), \|x\| \le 1 \right\} < 1 \text{ for some } t > 0, \tag{12.9}$$

then the operators $A + P$ and $A + (A - \lambda I)P(A - \lambda I)^{-1}$ are generators of C_0-cosine operator functions. Moreover, the C_0-cosine function $C_1(\cdot)$ generated by $A+P$ satisfies $\|C_1(t) - C(t, A)\| = O(\theta_P(t))$ $(t \to 0^+)$.

Proof. We may assume that A is invertible, so that $A + P = (I + PA^{-1})A$. We set $B = PA^{-1}$. Then

$$\int_0^t \|BS(s)Ax\| ds \le \int_0^t \|PS(t)x\| ds$$

for all $x \in D(A)$. Hence (12.9) implies $\delta_B(t) \le \theta_P(t) < 1$ for some $t > 0$, and the conclusion follows from Theorem 12.3.1.

From Theorem 12.2.2, we can deduce the following perturbation theorem of Watanabe ([287, Theorem 2]).

Corollary 12.3.1 ([240]). *Let A be the generator of a C_0-cosine operator function $C(\cdot, A)$ on E. If $P \in B(E^1, E)$, then $A + P$ and $A + (A - \lambda I)P(A - \lambda I)^{-1}$ $(\lambda > \omega)$ are generators of C_0-cosine operator functions. Moreover, the cosine function $C_1(\cdot)$ generated by $A + P$ satisfies $\|C_1(t) - C(t, A)\| = O(t)$ $(t \to 0^+)$.*

Proof. It is proved in [255] that $P \in B(E^1, E)$ implies (12.8). To show (12.9), let $x \in D(A)$. Then for $t \in [0, 1]$, we have

$$\|PS(t, A)x\| \le \|P\|_{B(E^1,E)} \|S(t, A)x\|_E \le \|P\|_{B(E^1,E)} \big[\|S(t, A)x\| + \sup_{0 \le \eta \le 1} \|AS(\eta, A)S(t, A)x\|\big] \le K\|x\|.$$

Therefore, $\theta_P(t) = O(t)$ $(t \to 0^+)$, and hence the conclusion follows from Theorem 12.3.2.

From Theorem 12.3.2, we can also deduce the following corollary: when $a = \infty$, it is Theorem A in [262] (see also [255, Corollary 2.1]), and when $a < \infty$, it is Corollary 2.2 in [255], which contains Theorem 3.2 in [262].

Corollary 12.3.2 ([240]). *Let A be the generator of a C_0-cosine operator function on E. Let P be an operator satisfying conditions* (12.8)*, and let*

$$L(\lambda) := \sup\Big\{ \int_0^a e^{-\lambda s} \|PS(s, A)x\| ds;\ x \in D(A), \|x\| \le 1 \Big\} < \infty \tag{12.10}$$

for some $a \in (0, \infty]$ and $\lambda > \omega$. Let $L(\infty) := \lim_{\lambda \to \infty} L(\lambda)$. Then for each ε with $|\varepsilon| < L(\infty)^{-1}$, $A + \varepsilon P$ and $A + \varepsilon(A - \lambda I)P(A - \lambda I)^{-1}$ $(\lambda > \omega)$ are generators of C_0-cosine operator functions.

Proof. Choose numbers $0 < \mu < \mu_1 < \mu_2 < 1$ such that $|\varepsilon| = \mu L(\infty)^{-1}$. Fix λ so large that $L(\lambda)/L(\infty) < \frac{\mu_1}{\mu}$, and then fix $t \in (0, a)$ so small that $e^{\lambda t} < \frac{\mu_2}{\mu_1}$. Then for all $x \in D(A)$ with $\|x\| \le 1$, we have

$$\int_0^t \|\varepsilon PS(s, A)x\| ds \le |\varepsilon| e^{\lambda t} \int_0^t e^{-\lambda s} \|PS(s, A)x\| ds \le |\varepsilon| e^{\lambda t} L(\lambda) = e^{\lambda t} \mu L(\lambda)/L(\infty) \le \frac{\mu_2}{\mu_1} \mu \frac{\mu_1}{\mu} = \mu_2 < 1,$$

i.e., $\theta_{\varepsilon P}(t) < 1$. Now the conclusion follows from Theorem 12.3.2.

From this corollary, Shimizu and Miyadera were able to deduce a generalization ([262, Corollary 2.2]) of the perturbation theorem of Fattorini [138] and Travis and Webb [275]. The latter theorem states that if a closed operator P satisfies $D(A) \subset D(P)$ and $PS(\cdot, A)x \in C([0,1]; E)$ for every $x \in E$, then $A + P$ is the generator of a C_0-cosine operator function. This is also an immediate consequence of Theorem 12.3.2, since it is clear that $\theta_P(t) = O(t)$ $(t \to 0^+)$.

Next, we consider mixed-type perturbations induced by a C_0-family of multiplicative perturbations and a C_0-family of additive perturbations. Thus, the following two theorems follow immediately from Theorems 12.2.1, 12.3.1, and Corollary 12.3.1.

Theorem 12.3.3 ([240]). *If a C_0-family of multiplicative perturbations $F(\cdot)$ for $C(\cdot, A)$ is locally of bounded semivariation and if* $\mathrm{SV}(F(\cdot), t) = o(1)$ $(t \to 0^+)$, *then the operator $A_1 := A(I - \lambda\hat{F}(\lambda)) + \lambda^3\hat{F}(\lambda)$, $\lambda > \omega$, is the infinitesimal generator of some cosine operator function $C_1(\cdot)$.*

Theorem 12.3.4 ([240]). *If a C_0-family of additive perturbations $G(\cdot)$ for $C(\cdot, A)$ is locally of bounded strong variation and if $\beta(F(\cdot), t) = o(1)$ $(t \to 0^+)$, then the operator $A_2 := (I - \lambda\hat{G}(\lambda))A + \lambda^3\hat{G}(\lambda)$, $\lambda > \omega$, is the infinitesimal generator of some C_0-cosine operator function $C_2(\cdot)$.*

12.4. Comparison of Cosine Operator Functions

In this section, we give some characterizations of the property that $\|C_1(t, A_1) - C(t, A)\| = O(t^2)$ $(t \to 0^+)$.

Theorem 12.4.1 ([240]). *Let $C(\cdot, A)$ be a C_0-cosine operator function with a generator A, and let A_1 be a linear operator. The following statements are equivalent:*

(i) *A_1 generates a C_0-cosine operator function $C_1(\cdot, A_1)$ that satisfies*

$$\|C_1(t, A_1) - C(t, A)\| = O(t^2) \quad (t \to 0^+);$$

(ii) *there exists $B \in B(E, \mathrm{Fav}_{C(\cdot,A)})$ such that $A_1 = A(I - B) + \lambda^2 B$ for some $\lambda > \omega$;*

(iii) *there exists $B \in B(E)$ such that the function $F(\cdot) \equiv F_{B,\lambda}(\cdot)$ defined in* (12.3) *is square Lipschitz continuous and $A_1 = A(I - \lambda\hat{F}(\lambda)) + \lambda^3\hat{F}(\lambda)$;*

(iv) *A_1 generates a C_0-cosine operator function $C(\cdot, A_1)$, $D(A_1^*) = D(A^*)$, and $A_1^* - A^*$ is a bounded operator acting from $D(A^*)$ to E^*;*

(v) *A_1 generates a C_0-cosine operator function $C(\cdot, A_1)$ and*

$$\|(\lambda^2 I - A_1)^{-1} - (\lambda^2 I - A)^{-1}\| = O(\lambda^{-4}) \quad (\lambda \to \infty).$$

Proof. The proof of (i) $\Rightarrow$ (ii) is similar to the proof for C_0-semigroups. Let B be the operator defined by $Bx := x - (\lambda^2 - A)^{-1}(\lambda^2 - A_1)x$ for $x \in D(A_1)$. Since for all $x \in D(A_1)$,

$$\begin{aligned}
\|Bx\| &= \lim_{\eta\to 0}\left\|x - \lambda^2(\lambda^2 I - A)^{-1}x + \frac{2}{\eta^2}(\lambda^2 I - A)^{-1}(C(\eta, A_1)x - x)\right\| \\
&= \lim_{\eta\to 0}\Big\|x - \lambda^2(\lambda^2 I - A)^{-1}x + \frac{2}{\eta^2}(\lambda^2 I - A)^{-1}(C(\eta, A)x - x) \\
&- \frac{2}{\eta^2}(\lambda^2 I - A)^{-1}(C(\eta, A)x - C(\eta, A_1)x)\Big\| \\
&\le \lim_{\eta\to 0}\|(\lambda^2 I - A)^{-1}\|\ \frac{2}{\eta^2}\ \|C(\eta, A) - C(\eta, A_1)\|\ \|x\| \le K\|(\lambda^2 I - A)^{-1}\|\ \|x\|,
\end{aligned}$$

where B is bounded and can be extended to a bounded operator (still denoted by B) on the whole space E. To show that B maps E to $\mathrm{Fav}_{C(\cdot,A)}$ continuously, we set $y = Bx$ for $x \in D(A_1)$. Then

$$\begin{aligned}
\limsup_{\eta\to 0}\frac{2}{\eta^2}\|C(\eta, A)y - y\| &\le \limsup_{\eta\to 0}\frac{2}{\eta^2}\big(\|(\lambda^2\eta^2 - C(\eta, A) + I)(x + y) - (\lambda^2\eta^2 - C_1(\eta, A_1) + I)x\| \\
&+\|C(\eta, A)x - C_1(\eta, A_1)x\|\big) + 2\lambda^2\|y\| \le K\|x\| + 2\lambda^2\|y\| \le K(1 + 2\lambda^2\|(\lambda^2 - A)^{-1}\|)\|x\|,
\end{aligned}$$

so that $|Bx|_{\mathrm{Fav}_{C(\cdot,A)}} \le K(1 + (1 + 2\lambda^2)\|(\lambda^2 - A)^{-1}\|)\|x\|$ for all $x \in D(A_1)$ (and hence for all $x \in E$). From $(\lambda^2 - A)(x - Bx) = (\lambda^2 - A_1)x$, we have $A_1 x = A(I - B)x + \lambda^2 Bx$ for all $x \in D(A_1)$, i.e., $A_1 \subseteq A(I-B)+\lambda^2 B$. Since $\mathrm{Fav}_{C(\cdot,A)}$ satisfies Condition (Z), Corollary 12.1.1 implies that $A(I-B)+\lambda^2 B$ is the generator of a C_0-cosine operator function and hence coincides with A_1.

Taking the Laplace transform of $F(\cdot)$, we obtain $\hat{F}(\mu) = \lambda^2\mu^{-1}(\mu^2 - A)^{-1}B - \mu(\mu^2 - A)^{-1}B + \mu^{-1}B$, so that $B = \lambda\hat{F}(\lambda)$. Further, using (12.3), for all $x \in E$, we have

$$\big|\|F(t)x\| - \|(C(t, A) - I)Bx\|\big| \le \|\lambda^2\int_0^t S(s, A)Bxds\| = O(t^2)\ (t \to 0^+),$$

which implies that $B \in B(E, \mathrm{Fav}_{C(\cdot,A)})$ iff $\|F(t)\| = O(t^2)\ (t \to 0^+)$. Hence (ii) and (iii) are equivalent.

(iii) $\Rightarrow$ (i). In view of Theorem 12.3.3, we need only to show that if $F(t) = O(t^2)\ (t \to 0^+)$, then $\mathrm{Var}(F(\cdot), t) = O(t^2)\ (t \to 0^+)$. But, because of (12.3), this is equivalent to showing that $\|(C(t, A)-I)B\| = O(t^2)\ (t \to 0^+)$ implies $\mathrm{Var}(C(\cdot, A)B, t) = O(t^2)\ (t \to 0^+)$. Therefore, we suppose that $\|(C(s, A)-I)B\| \le Ks^2$ for $0 \le s \le \tau$. For any subdivision $\{t_0, t_1, \cdots, t_n\}$ of $[0, t] \subseteq [0, 1]$ with $h_i = t_i - t_{i-1} \le \tau$, let n_i be the largest integer such that $n_i h_i \le t_i$. One has

$$C(t_i, A) - C(t_{i-1}, A) = C(t_{i-1}, A) - C(t_{i-1,A} - C(t_i - t_{i-1})) + 2C(t_{i-1}, A)(C(t_i - t_{i-1}, A) - I),$$

and, therefore,

$$\begin{aligned}
\|C(t_i, A) - C(t_{i-1}, A)\| &\le \|C(t_i - n_i h_i, A) - C((n_i + 1)h_i - t_i, A)\| + 2KMe^{\omega t}n_i h_i^2 \\
&\le 2KMe^{\omega t}(n_i + 2)h_i^2 \le 4KMe^{\omega t}t h_i.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}\|C(t_i,A)-C(t_{i-1},A)\|\le 4KMe^{\omega t}t\sum_{i=1}^{n}h_i\le 4KMe^{\omega t}t^2.$$

Hence $\mathrm{Var}(C(\cdot,A)B,t)=O(t^2)\,(t\to 0^+)$.

(i) $\Leftrightarrow$ (iv) is proved in ([238, Theorem 3.5]).

To prove (iv) $\Rightarrow$ (v), we write

$$\begin{aligned}\|(\lambda^2I-A_1)^{-1}-(\lambda^2I-A)^{-1}\| &= \|(\lambda^2I-A_1^*)^{-1}(A_1^*-A^*)(\lambda^2I-A^*)^{-1}\|\\ &\le \|(\lambda^2I-A_1)^{-1}\|\|A_1^*-A^*\|\|(\lambda^2I-A)^{-1}\|=O\left(\frac{1}{\lambda^4}\right).\end{aligned}$$

Finally, to prove (v) $\Rightarrow$ (i), we write (see [238, Theorem 3.9])

$$\begin{aligned}\|C(t,A_1)x-C(t,A)x\| &= \lim_{\lambda\to\infty}\lambda^4\|(\lambda^2I-A_1)^{-1}(C(t,A_1)-C(t,A))(\lambda^2I-A)^{-1}x\|\\ &\le \lim_{\lambda\to\infty}\|\int_0^t S(t-s,A_1)\lambda^4((\lambda^2I-A_1)^{-1}-(\lambda^2I-A)^{-1})C(s,A)xds\|\le Kt^2\|x\|.\end{aligned}$$

Hence mixed type perturbations of the form $A_1=A(I-B)+\lambda^2B$ with $B\in B(E,\mathrm{Fav}_{C(\cdot,A)})$ characterize those C_0-cosine functions $C_1(\cdot,A_1)$ which satisfy $\|C(t,A_1)-C(t,A)\|=O(t^2)$ $(t\to 0^+)$. In this case, although $D(A_1^*)=D(A^*)$, the domain of A_1 may not contain the domain of A. What kind of C_0-cosine operator functions $C(\cdot,A_1)$ have the property that $D(A)\subseteq D(A_1)$ and $\|C(t,A_1)-C(t,A)\|=O(t^2)\,(t\to 0^+)$? It is clear that additive perturbations of A by bounded operators generate cosine functions with this property.

Theorem 12.4.2 ([240]). *Let $C(\cdot,A)$ be a cosine operator function with the generator A. For any operator A_1, the following statements are equivalent*:

(i) *$D(A)\subseteq D(A_1)$, and A_1 generates a C_0-cosine operator function $C_1(\cdot)$ such that*

$$\|C_1(t,A_1)-C(t,A)\|=O(t^2)\,(t\to 0^+);$$

(ii) *there exists an operator $B\in B(E)$ such that $R(B^*)\subseteq \mathrm{Fav}_{C^*(\cdot,A)}$ and $A_1=(I-B)A+\lambda^2B$ for some $\lambda>\omega$;*

(iii) *there exists $B\in B(E)$ such that the function $G(\cdot)\equiv G_{B,\lambda}(\cdot)$ defined in* (12.4) *is square Lipschitz continuous at* 0 *and $A_1=(I-\lambda\hat{G}(\lambda))A+\lambda^3\hat{G}(\lambda)$;*

(iv) *$A_1=A+Q$ for some $Q\in B(E)$.*

Proof. (iv) $\Rightarrow$ (1) is obvious. We first prove (i) $\Rightarrow$ (ii)+(iv). Since $D(A)\subseteq D(A_1)$, we can define the bounded operator $B:=I-(\lambda^2I-A_1)(\lambda^2I-A)^{-1}=(A_1-A)(\lambda^2I-A)^{-1}$ for $\lambda>\omega$. Then we have

$A_1 x = (I - B)Ax + \lambda^2 Bx$ for $x \in D(A)$. Using (12.4), we can write

$$G(t)x = (A_1 - A)\int_0^t S(s,A)x\,ds,\ \ x \in E,\ t \geq 0.$$

Since $\|(A_1 - A)x\| \leq \overline{\lim_{t\to 0^+}}\frac{2}{t^2}\|(C(t,A_1) - C(t,A))x\| \leq K\|x\|$ for all $x \in D(A)$, the operator $A_1 - A$ has a bounded extension $Q \in B(E)$. Thus (iv) holds. This also implies

$$\|G(t)x\| \leq \mathrm{Var}(G(\cdot)x,t) \leq \|Q\|\int_0^t \|S(s,A_1)x\|ds \leq \|Q\|Me^{\omega t}t^2\|x\|,\ x \in E,$$

so that $\|G(t)\| \leq \beta(G_{B,\lambda}(t) = O(t^2)$. It follows from Theorems 12.2.1 and 12.1.3 that $(I - B)A + \lambda^2 B$ is a generator of cosine function, and hence it coincides with A_1. Further, using (12.4) for all $x^* \in E^*$, we have

$$\big|\|G^*(t)x^*\| - \|(C^*(t,A) - I^*)B^*x^*\|\big| \leq \|\lambda^2\int_0^t S^*(s,A)B^*x^*ds\| = O(t^2)\ \ (t \to 0^+).$$

Hence $\|G(t)\| = O(t^2)$ $(t \to 0^+)$ iff $R(B^*) \subseteq \mathrm{Fav}_{C^*(\cdot,A)}$. In particular, this completes the proof of (ii).

To prove (ii) $\Leftrightarrow$ (iii), it remains to show that $B = \lambda\hat{G}(\lambda)$. This can be done by taking Laplace transform of $G_{B,\lambda}(\cdot)$ in (12.4).

(iii) $\Rightarrow$ (i). It suffices to show that $R(B^*) \subseteq \mathrm{Fav}_{C^*(\cdot,A)}$ implies the boundedness of BA. Indeed, for all $x \in D(A)$ with $\|x\| \leq 1$ and for all $x^* \in E^*$, we have

$$|\langle BAx, x^*\rangle| = |\lim_{t\to 0} 2t^{-2}\langle (C(t,A) - I)x, B^*x^*\rangle| \leq \overline{\lim_{t\to 0}}2t^{-2}\|(C^*(t,A) - I^*)B^*x^*\|.$$

The uniform boundedness principle implies that $\{BAx : x \in D(A),\ \|x\| \leq 1\}$ is bounded. Hence the operator BA is bounded on $D(A)$.

12.5. Preservation of Properties under Additive Perturbations

This section contains, as a rule, known facts. However, they have appeared historically first and, moreover, in this section some useful relations are formulated explicitly.

Proposition 12.5.1 ([222, 287]). *Let $A \in \mathcal{C}(M,\omega)$, and let $B \in B(E)$. Then the operator $A+B$ generates a C_0-cosine operator function and $\|C(t,A+B) - C(t,A)\| \to 0$ as $\|B\| \to 0$ uniformly on any compact set in $\mathbb{R}$.*

Proposition 12.5.2 ([221]). *Under conditions of Proposition 12.5.1, if $\tilde{\omega} > \omega + \frac{M}{\omega}\|B\|$, then there exists a number $\tilde{M} = \tilde{M}(\omega)$ such that*

$$\|C(t,A+B)\| \leq \tilde{M}e^{\tilde{\omega}|t|}, \quad t \in \mathbb{R}. \tag{12.11}$$

Proposition 12.5.3 ([132]). *Let $A \in \mathcal{C}(M,\omega)$. Then for $x \in E$,*

$$C(t, \zeta^2 I + A)x = C(t, A)x + \zeta t \int_0^t \frac{I_1(\zeta\sqrt{t^2 - s^2})}{\sqrt{t^2 - s^2}} C(s, A)x \, ds, \tag{12.12}$$

where I_1 is the Bessel function and

$$\|C(t, A + \zeta^2 I)\| \leq M \cosh(\sqrt{\zeta^2 + \omega^2}\, t) \quad \textit{for} \quad \zeta \in \mathbb{C}.$$

In [132,265,266], other more precise estimates of the expression $C(t, A \pm \zeta^2 I)$ are presented for Banach and Hilbert spaces.

Proposition 12.5.4 ([255, 288]). *Let $A \in \mathcal{C}(M,\omega)$, and let $D(A) \subseteq D(G)$. If there exist $\omega' \geq 0$ and $M' \geq 1$ such that $\rho(A)$ contains the set $\{z : z > \omega'\}$ and the function $G(z^2 I - A)^{-1}$ is infinitely many times differentiable, and, moreover,*

$$\frac{1}{n!} \left\| (z - \omega')^{n+1} \Big(\frac{d}{dz}\Big)^n \Big(G(z^2 I - A)^{-1}\Big) x \right\| \leq M' \|x\|$$

for $x \in E$ and any $n \in \mathbb{N}$ and $z > \omega'$, then $A + G$ generates a C_0-cosine operator function.

Proposition 12.5.5 ([144]). *If $A, G \in \mathcal{C}(M,\omega)$, then the operator $A+G$ (or its closure) may not generate a C_0-cosine operator function in general, even in the case where $C(\cdot, A)$ and $C(\cdot, G)$ commute. However, we always have $\overline{A + B} \in \mathcal{H}(\omega, \pi/2)$.*

Proposition 12.5.6 ([21]). *Let $A, G \in \mathcal{C}(M,\omega)$, and let $D_1 := D(A) \cap D(G)$ be dense in E. Then on D_1, the following "generalized" cosine function (in the sense of fulfillment of conditions* (i)–(ii) *of Definition* 2.3.1) *is defined:*

$$\tilde{C}(t, A + G)x = C(t, A)x + \frac{t^2}{2} \int_0^1 j_1(t\sqrt{1 - s^2}, A) C(ts, G)x \, ds,$$

where $j_1(t, A) := \frac{4}{\pi} \int\limits_0^1 \sqrt{1 - s^2} C(ts, A)x \, ds$, $x \in D_1$. If $A + G$ generates a C_0-cosine operator function $C(t, A + G)$, then $C(t, A + G) = \tilde{C}(t, A + G)$, $t \in \mathbb{R}$.

Proposition 12.5.7 ([275]). *Let $A \in \mathcal{C}(M,\omega)$, and let the operator $G \in \mathcal{C}(E)$ be such that*

(i) *for the C_0-sine operator valued function, $\mathcal{R}(S(t, A)) \subseteq D(G)$ for all $t \in \mathbb{R}$;*

(ii) *the function $GS(t, A)$ is strongly continuous in $t \in \mathbb{R}$.*

Then the operator $A + G$ generates a C_0-cosine operator function. Moreover,

$$C(t, A + G)x = \sum_{k=0}^{\infty} \hat{C}_k(t) \quad \textit{and} \quad S(t, A + G)x = \sum_{k=0}^{\infty} \hat{S}_k(t), \tag{12.13}$$

where $\hat{C}_0(t) := C(t, A)$,

$$\hat{C}_k(t) := \int_0^t C(t-s, A)G\hat{S}_{k-1}(s)\, ds,$$

and $\hat{S}_0(t) := S(t, A)$,

$$\hat{S}_k(t) := \int_0^t S(t-s, A)G\hat{S}_{k-1}(s)\, ds,$$

and series (12.13) *converges absolutely in* $B(E)$.

Proposition 12.5.8 ([275]). *Under conditions of Proposition* 12.5.7, *we have the relation*

$$(\lambda I - A - G)^{-1} = (\lambda I - A)^{-1} \sum_{k=0}^{\infty} (G(\lambda I - A)^{-1})^k.$$

Proposition 12.5.9 ([273]). *Under conditions of Proposition* 12.5.7, *if the* C_0*-sine operator function* $S(\cdot, A)$ *is compact, then the* C_0*-sine operator function* $S(\cdot, A+G)$ *is also compact.*

Proposition 12.5.10 ([269]). *Let* $A_1, A_2 \in \mathcal{C}(M, \omega)$, *and let* $D(A_1) \subseteq D(A_2)$. *Then*

$$C(t, A_1)x - C(t, A_2)x = \int_0^t S(t-s, A_2)(A_1 - A_2)S(s, A_1)x\, ds$$

for all $x \in D(A_1)$.

Theorem 12.5.1 ([269, 273]). *Let* $A \in \mathcal{C}(M, \omega)$, *and let* $G \in \mathcal{C}(E)$ *be such that*

(i) $D(A) \subseteq D(G)$;

(ii) *there exists a continuous function* $K(t)$ *having the property*

$$\|GS(t, A)x\| \leq K(t)\|x\| \quad \text{for all} \quad x \in D(A).$$

Then the operator $A + G$ *generates a* C_0*-cosine operator function.*

Proposition 12.5.11 ([273]). *Let an operator* $A \in \mathcal{C}(M, \omega)$ *satisfy Condition* (F) *with an operator* $G \in \mathcal{C}(E)$, *and, moreover, for a certain operator* $Q \in \mathcal{C}(E)$, *let the condition* $D(G) \subseteq D(Q)$ *hold. Then* $A + Q$ *generates a* C_0*-cosine operator function.*

Proposition 12.5.12 ([273]). *In Proposition* 12.5.11, *let the condition of inclusion of domains be replaced by the condition* $D(A) \subseteq D(Q)$. *If the operator* Q *is* G*-bounded, then the operator* $A + Q$ *generates a* C_0*-cosine operator function.*

Proposition 12.5.13 ([262, 270]). *Let* $A \in \mathcal{C}(M, \omega)$, $G \in \mathfrak{M}(C(t, A))$, $G(\lambda I - A)^{-1} \in B(E)$, *and let* $|\varepsilon| < K_\infty^{-1}$. *Then the operator* $A + \varepsilon G$ *generates a* C_0*-cosine operator function,* $\lim_{\varepsilon \to 0} \|C(t, A + \varepsilon G) - C(t, A)\| = 0$

uniformly on any compact set from $\mathbb{R}$, *and*

$$C(t, A + \varepsilon G) = \sum_{k=0}^{\infty} \varepsilon^k \overline{C}_k(t),$$

where $\overline{C}_0(t) := C(t, A)$; $C_k(t) := \int_0^t \overline{C}_{k-1}(t-s)GS(s, A)x\,ds$, $x \in D(A)$, $\overline{C}_k(t)$ *is a continuous extension of* $C_k(t)$ *to the whole* E *and* $\mathfrak{M}$ *is the class of perturbations in the sense of Miyadera.*

Proposition 12.5.14 ([208]). *Let* C_0*-cosine operator functions* $C(t, A_1)$ *and* $C(t, A_2)$ *be such that* $\|C(t, A_1)\| \le Me^{\omega t}$ *and* $\|C(t, A_2)\| \le Ne^{\nu t}$ *for* $t \in \overline{\mathbb{R}}$ *and* $\|(A_1 - A_2)x\| \le a\|x\| + b\|A_1 x\|$ *for* $x \in D(A_1)$. *Then for* $z > \omega$, *we have*

$$\left\|\Big(C(t, A_1) - C(t, A_2)\Big)(z^2 I - A_1)^{-1}\right\| \le \begin{cases} \dfrac{MN}{2\nu} Qt\sinh(\omega t), \\ \dfrac{MN}{\omega^2 - \nu^2} Q\Big(\cosh(\omega t) - \cosh(\nu t)\Big), \end{cases}$$

where $Q := (1 + M)b + \dfrac{(a + b\omega^2)M}{z^2 - \omega^2}$.

Proposition 12.5.15 ([208]). *Under conditions of Proposition* 12.5.14, *if* $a, b \to 0$, *then*

$$s\text{-}\lim_{a,b\to 0} C(t, A_2)x = C(t, A_1)x \quad \textit{for} \quad x \in D(A_1).$$

Proposition 12.5.16 ([259]). *Let* $K(t, A)$ *be a* w^**-continuous cosine operator function, and let* B *be a* w^**-*w^**-continuous operator on* E^*. *Then* $A + B$ *is the* w^**-generator of a* w^**-continuous cosine-function* $K(t, A + B)$ *and* $\lim_{\|B\|\to 0} \|K(t, A + B) - K(t, A)\| = 0$ *uniformly on any compact set* $t \in [0, T]$.

12.6. An Integral Operator on $L^p([0, T]; E)$

For C_0-groups of operators and C_0-cosine operator functions, we can formulate specific perturbation theorems, which assume certain "hyperbolicity" conditions.

Let $J \subseteq \mathbb{R}$ be a certain interval. Denote by $\Sigma(J; E)$ the vector space of all linear combinations of mappings of the form $\chi_j x$, where $x \in E$ and χ_j is the characteristic function of the interval $\mathcal{T} \subseteq J$, i.e., $\Sigma(J; E)$ is the space of step functions.

Let $\{L(t)\}_{t=-\infty}^{\infty}$ be a strongly continuous family of bounded operators on E, and let $A \in \mathcal{C}(E)$. Assume that the following hypotheses hold:

H1. for each $x \in E$ and $t \in \mathbb{R}$, the integral $\int_0^t L(s)x\,ds \in D(A)$ and the mapping $t \to A\int_0^t L(s)x\,ds$ is continuous as a mapping from $\mathbb{R}$ into E;

H2. there exist a subset $D^{\odot} \subseteq D(A^*)$ and a constant $M \ge 1$ such that

(a) for any $\phi^{\odot} \in D^{\odot}$, the mapping $t \to L^*(t)A^*\phi^{\odot}$ is continuous as a mapping from $\mathbb{R}$ into E^*;

(b) for each $x \in E$, there exists $\phi^{\odot} \in D^{\odot}$ such that $\|\phi^{\odot}\| \le M$ and $\|x\| = \langle x, \phi^{\odot}\rangle$.

By H1, for any $f(\cdot) \in \Sigma(\mathbb{R}; E)$, the mapping $t \to A \int\limits_0^t L(t-s)f(s)\,ds$ is continuous as a mapping from $\mathbb{R}$ into E, and we can define the operator $K : \Sigma(\mathbb{R}; E) \to C(\mathbb{R}; E)$ by the formula

$$(Kf)(t) := A \int_0^t L(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \quad f(\cdot) \in \Sigma(J; E).$$

By H2, for each $g^{\odot}(\cdot) \in \Sigma(\mathbb{R}; D^{\odot})$, we can analogously define $K^{\odot} : \Sigma(\mathbb{R}; D^{\odot}) \to C(\mathbb{R}; E^*)$ by the formula

$$(K^{\odot}g^{\odot})(s) := \int_s^{\infty} L^*(t-s)A^*g^{\odot}(t)\,dt, \quad s \in \mathbb{R}, \quad g^{\odot}(\cdot) \in \Sigma(\mathbb{R}; D^{\odot}).$$

Let $T > 0$ be finite. Then the operators K and $K^{\odot}$ induce the operators

$$(K_T f)(t) := A \int_0^t L(t-s)f(s)\,ds, \quad t \in [0,T], \quad f(\cdot) \in \Sigma(J; E) \tag{12.14}$$

and

$$(K_T^{\odot}g^{\odot})(s) := \int_s^T L^*(t-s)A^*g^{\odot}(s)\,ds, \quad s \in [0,T]; \quad g^{\odot}(\cdot) \in \Sigma(J; D^{\odot}).$$

Let $p, q \in \overline{\mathbb{R}}_+$, and let $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For $f(\cdot) \in L^p(J; E)$ and $g^*(\cdot) \in L^q(J; E^*)$, we set

$$\langle\langle f, g \rangle\rangle = \int_J \langle f(s), g^*(s) \rangle\,ds,$$

and by the Fubini theorem, we have

$$\langle\langle K_T f, g^{\odot} \rangle\rangle = \langle\langle f, K_T^{\odot} g^{\odot} \rangle\rangle, \quad f(\cdot) \in \Sigma([0;T]; E), \; g^{\odot}(\cdot) \in \Sigma([0;T]; D^{\odot}).$$

Now we note that if

$$|||K_T|||_{p_1,p_2} := \sup\left\{ \|K_T f\|_{L^{p_2}([0;T];E)} : f(\cdot) \in \Sigma([0;T]; E), \|f\|_{L^{p_1}([0,T];E)} = 1 \right\} < \infty,$$

then by the denseness of $\Sigma([0;T]; E)$ in $L^{p_1}([0,T]; E)$ and the closedness of the operator A, for any $f(\cdot) \in L^{p_1}([0,T]; E)$, the element $K_T f(\cdot) \in L^{p_2}([0,T]; E)$ and is defined for almost all $t \in [0,T]$ by expression (12.14). We call attention to that if $p_2 = \infty$, then the range of K_T, in fact, lies in $C([0,T]; E)$, and $K_T : L^{p_1}([0;T]; E) \to C([0;T]; E)$ is continuous. Analogous arguments lead to the continuity of the mapping $K_T^{\odot} : L^{q_2}([0;T]; E^{\odot}) \to C([0;T]; E^*)$, where $E^{\odot}$ is the strong closure of $D^{\odot}$. Moreover, using H2 (b), we obtain

$$|||K_T^{\odot}|||_{q_2,q_1} \leq |||K_T|||_{p_1,p_2} \leq M|||K_T^{\odot}|||_{q_2,q_1}, \tag{12.15}$$

where $\dfrac{1}{p_1} + \dfrac{1}{q_1} = 1$, $\dfrac{1}{p_2} + \dfrac{1}{q_2} = 1$, and

$$|||K_T^{\odot}|||_{q_2,q_1} := \sup\left\{ ||K_T^{\odot}g^{\odot}||_{L^{q_1}([0;T];E^*)} : g^{\odot} \in \Sigma([0;T]; D^{\odot}), \|g^{\odot}\|_{L^{q_2}([0;T];E^*)} = 1 \right\}.$$

12.7. Lifting Theorem for C_0-Groups

Recall that for a generator of a C_0-semigroup $\exp(\cdot A)$, the domain $D(A^{\odot})$ is a w^*-dense set in E^*, and $A^{\odot}$ is a closed operator on $E^{\odot} = \overline{D(A^{\odot})}$.

In this section, we consider the operator K_T defined by the formula

$$(K_T f)(t) := A \int_0^t \exp((t-s)A)Bf(s)\, ds, \quad t \in [0,T], \tag{12.16}$$

where $B \in B(E)$.

Theorem 12.7.1 ([232]). *Let $A \in \mathcal{GR}(M,\omega)$, and let $K_T \in B\Big(L^{p_1}([0,T];E), L^{p_2}([0,T];E)\Big)$ for certain $p_1, p_2 \in \overline{\mathbb{R}}_+$ and $T \in \mathbb{R}_+$. Then $K_T \in B\Big(L^{p_1}([0,T];E), C([0,T];E)\Big)$ and there exists a constant $C > 0$ such that*

$$\|K_T f\|_{C([0,\tau];E)} \le C\|f\|_{L^{p_1}([0,\tau];E)} \quad \textit{for any} \quad \tau \in [0,T]. \tag{12.17}$$

Definition 12.7.1. We say that the operator K_T for the C_0-group $\exp(\cdot A)$ satisfies Condition HG if there exist constants $C, \tau > 0$ and $T > 0$ such that for any $x \in E$, there exists a function $h_x(\cdot) \in C([-\tau,T];E)$ having the properties

(**HG**1) $\|h_x(\cdot)\|_{C([-\tau,T];E)} \le C\|x\|_E$;

(**HG**2) $Bh_x(t) = \exp(tA)Bx, \quad -\tau \le t \le T-\tau$.

Condition HG hor C_0-groups holds, e.g., in the following two cases:

(i) B commutes with $\exp(\cdot A)$; in this case, $h_x(t) = \exp(tA)x$, $-\tau \le t \le T$;

(ii) B has a bounded inverse; in this case, $h_x(t) = B^{-1}\exp(tA)Bx$, $t \in [-\tau, T]$.

Theorem 12.7.2 ([232]). *Let conditions of Theorem* 12.7.1 *hold, and, additionally, let Condition* HG *hold. Then $K_T \in B\Big(L^1([0,T];E), C([0,T];E)\Big)$ and there exists a constant $C > 0$ such that $\|K_T f\|_{C([0,\tau];E)} \le C\|f\|_{L^1([0,\tau];E)}$ for any $0 \le \tau \le T$.*

Corollary 12.7.1 ([232]). *Let conditions of Theorem* 12.7.1 *hold. Then there exist constants $L, \alpha > 0$ such that* $\mathrm{SV}(K_T, T) \le Le^{\alpha T}T^{1/p_1}$, $T > 0$.

Corollary 12.7.2 ([232]). *Let $E = H$ be Hilbert, and let conditions of Theorem* 12.7.1 *hold. Then there exist constants $L, \alpha > 0$ such that* $\mathrm{SV}(K_T, T) \le Le^{\alpha T}T^{1/2}$, $T > 0$.

12.8. Lifting Theorem for C_0-Cosine Operator Functions

Let E be a Banach space, and let $A : D(A) \subset E \to E$ be a generator of a C_0-cosine operator function on E. Also, let us consider the adjoint family $\{C(\cdot, A)^*\}$. It is well known that it is also a cosine family of linear bounded operators on the dual space, which, however, can be not strongly continuous. Recall that the space $E^{\odot}$ is defined as

$$E^{\odot} = \{\, x^* \in E^* \,:\, s - \lim_{t \to 0} C(t, A)^* x^* = x^* \,\},$$

where the limit is understood in the strong topology of the space E^*.

On the other hand, C_0-cosine operator functions can be studied by using C_0-groups. More precisely, introducing the Kysinski space E^1, we can reduce the consideration to a C_0-group of operators $\{\exp(t\mathcal{A})\}_{t=-\infty}^{+\infty}$ on $E^1 \times E$ defined as

$$\exp(t\mathcal{A}) = \begin{pmatrix} C(t, A) & S(t, A) \\ AS(t, A) & C(t, A) \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{12.18}$$

Its generator is $\mathcal{A} : D(A) \times E^1 \subset E^1 \times E \to E^1 \times E$; it is given by $\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$.

Let $B \in B(E)$ be a linear continuous operator on E. It is easy to verify that the family $L(t) := S(t, A)B$, $t \in \mathbb{R}$, satisfies all the conditions presented in Sec. 12.6 with respect to $D^{\odot} = D(A^{\odot})$.

In this section, we consider the continuity of the corresponding convolution operator

$$(K_T f)(t) := A \int_0^t S(t - s, A) B f(s)\, ds, \qquad t \in [0, T] \subset \overline{\mathbb{R}}_+, \tag{12.19}$$

in $L^p([0, T]; E)$ norms, $p \in [1, +\infty]$.

Note that in the case of the C_0-cosine operator functions considered, one can show that $\|K_T\|_{B(L^{p_1}([0,T];E), L^{\infty}([0,T];E))}$ grows exponentially in T. This property is stated in the lifting theorem (Theorem 12.8.1).

Further, it is natural to try to find a new convolution operator $\mathcal{K}_{\mathcal{T}}$ constructed according to the group $\mathcal{U}$ that reduces the study of K_T to that of $\mathcal{K}_{\mathcal{T}}$. Therefore, the results that are true for C_0-groups can be directly extended to C_0-cosine operator functions. Consider the following operator bounded on $E^1 \times E$:

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

and the corresponding convolution operator

$$(\mathcal{K} h)(t) = \mathcal{A} \int_0^t \mathcal{U}(t - s) \mathcal{B} h(s)\, ds, \qquad t \in \mathbb{R}, \tag{12.20}$$

where $h = [g, f]^T \in \Sigma([0, T], E^1 \times E)$.

Also, for $T > 0$, we define the convolution operator $G_T : \Sigma([0, T], E) \to C([0, T], E^1)$ by the relation

$$(G_T f)(t) = \int_0^t C(t - s, A) B f(s)\, ds, \qquad f \in \Sigma([0, T], E), \quad 0 \le t \le T.$$

Further, we have

$$(\mathcal{K}_T h)(t) = [(G_T f)(t), (K_T f)(t)]^T, \qquad 0 \le t \le T,$$

for $h = [g, f]^T \in \Sigma([0,T], E^1 \times E)$.

Lemma 12.8.1 ([232])**.** *Let p_1, $p_2 \in [1, +\infty]$, and let $T > 0$.*

(i) *$\mathcal{K}_T$ is continuous as a mapping from $L^{p_1}([0,T], E^1 \times E)$ into $L^{p_2}([0,T], E^1 \times E)$ iff G_T is continuous as a mapping from $L^{p_1}([0,T]; E)$ into $L^{p_2}([0,T], E^1)$ and K_T is continuous from $L^{p_1}([0,T]; E)$ into $L^{p_2}([0,T]; E)$.*

(ii) *Let $T^* > T$. If K_{T^*} is continuous from $L^{p_1}([0,T^*], E)$ into $L^{p_2}([0,T^*], E)$ and if $p_2 = +\infty$, then $\mathcal{K}_T$ is continuous from $L^{p_1}([0,T], E^1 \times E)$ into $L^{p_2}([0,T], E^1 \times E)$.*

Assertion (ii) of the lemma with $T = T^*$ and $1 \le p_2 < +\infty$ also holds. This result is a consequence (see Corollary 12.8.2) of the main lifting theorem (Theorem 12.8.1).

Theorem 12.8.1 ([232])**.** *Assume that there exist T_0 and $p_1, p_2 \in [1, +\infty]$ such that $\|K_{T_0}\| \in B(L^{p_1}([0,T]; E), L^{p_2}([0,T]; E))$. Then there exist constants $L > 0$ and $\alpha > 0$ such that*

$$\|K_T\|_{B(L^{p_1}([0,T];E), C([0,T];E))} \le L e^{\alpha T}, \qquad T > 0. \tag{12.21}$$

Corollary 12.8.1 ([232])**.** *Assume that the conditions of Theorem 12.8.1 hold. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/p_1}, \qquad T \in \mathbb{R}.$$

Corollary 12.8.2 ([232])**.** *Let p_1, $p_2 \in [1, +\infty]$, and let $T > 0$. Then $\|K_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < +\infty$ iff $\|\mathcal{K}_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < +\infty$.*

Corollary 12.8.3 ([232])**.** *Assume that E is a Hilbert space and that the conditions of Theorem 12.8.1 hold. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/2}, \qquad T > 0.$$

In particular, setting $p_1 = p_2 = +\infty$ in the obtained assertion, we obtain the following interesting consequence.

Corollary 12.8.4 ([232])**.** *Assume that E is Hilbert and that $\mathrm{SV}(K_T, T_0) < +\infty$ for a certain $T_0 > 0$. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/2}, \qquad T \in \mathbb{R}_+.$$

In the same way as for C_0-groups, under certain additional conditions, we can strengthen the assertion of Theorem 12.8.1.

Definition 12.8.1. We say that the operator K_T in (12.19) satisfies Condition HC if there exist constants $C_0 > 0$ and $T_0 > 0$ such that for any $x \in E$, there exists a function $h_x \in L^\infty([0,T];E)$ having the properties
HC1 $\|h_x\|_{L^\infty([0,T];E)} \leq C\|x\|$
and
HC2 $Bh_x(t) = C(t-T,A)Bx,\ 0 \leq t \leq T.$

Note that at least in two cases, Condition HC holds; namely,
(i) when B commutes with $C(t,A)$; for $t \in \mathbb{R}$, we can set $h_x(t) = C(t-T,A)x$, $0 \leq t \leq T$, and
(ii) when B is invertible; we can set $h_x(t) = B^{-1}C(t-T,A)Bx$, $0 \leq t \leq T$.

Theorem 12.8.2 ([232]). *Assume that there exist $T_0 > 0$ and $p_1, p_2 \in [1,\infty]$ such that*

$$\|K_{T_0}\|_{B(L^{p_1}([0,T];E),L^{p_2}([0,T];E))} < \infty.$$

Also, assume that Condition HC *holds. Then there exist $L > 0, \alpha > 0$ such that*

$$\|K_T\|_{B(L^1([0,T];E),C([0,T];E))} < Le^{\alpha T}.$$

12.9. Perturbation of C_0-Groups of Operators

In this section, we use lifting theorems for studying multiplicative perturbations. Let $\exp(\cdot A)$ be a C_0-group of linear bounded operators on a Banach space E. Let $B \in B(E)$ be a certain linear bounded operator on E. Here we study the question whether the multiplicatively perturbed operator $A_m = A(I+B)$ defined on the natural domain $D(A_m) = \{x \in E : (I+B)x \in D(A)\}$ is an infinitesimal operator of another C_0-group under a condition on the operator K_T. First, assume that A_m is the infinitesimal operator of the C_0-group $\exp(\cdot A_m)$. It is easy to verify that for any $x \in E$, the following relation holds:

$$\exp(tA_m)x = \exp(tA)x + A\int_0^t \exp((t-s)A)B\exp(sA_m)x\,ds, \qquad t \in [0,T]. \tag{12.22}$$

Consider the convolution operator $K_T : \Sigma([0,T],E) \to C([0,T];E)$ defined in (12.16). Assume that there exists $T > 0$ such that $\mathrm{SV}(K_T,T) = \|K_T\|_{B(L^\infty([0,T];E),L^\infty([0,T];E))} < +\infty$. Then K_T can be considered as a bounded operator $K_T : C([0,T];E) \to C([0,T];E)$. Relation (12.22) means that for any $x \in E$, the mapping $f_{x,T} : [0,T] \to E$ defined by $f_{x,T}(t) = \exp(tA_m)x$ for $t \in [0,T]$ satisfies the condition

$$f_{x,T}(t) = \exp(tA)x + (K_T f_{x,T})(t), \qquad t \in [0,T].$$

The converse is also true. For a given $x \in E$, we study the solvability in $C([0,T];E)$ of the convolution equation

$$f(t) = \exp(tA)x + (K_T f)(t), \qquad 0 \le t \le T, \tag{12.23}$$

and then show that $f(t)$ can be considered as $\exp(tA_m)x$. As in Sec. 12.1, this approach is divided into two steps:

(Step 1) prove that there exists $T > 0$ such that K_T maps $C([0,T];E)$ into itself continuously and

(Step 2) prove that $\mathrm{SV}(K_T, T) = \|K_T\|_{B(L^\infty([0,T];E),L^\infty([0,T];E))} < 1$ for a certain sufficiently small $T > 0$.

If (Step 1) and (Step 2) hold, then by the contraction mapping principle, Eq. (12.23) is uniquely solvable. Therefore, following Sec. 12.1, we obtain that the perturbed operator A_m is the generator of a C_0-semigroup.

Theorem 12.9.1 ([232]). *Assume that there exist $T_0 > 0$, $1 \le p_1 < +\infty$, and $1 \le p_2 \le +\infty$ such that $K_T \in B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))$. Then there exist $L > 0$ and $\alpha > 0$ such that for any $T > 0$,*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/p_1}. \tag{12.24}$$

Moreover, the operator A_m generates a C_0-group and

$$\|\exp(tA) - \exp(tA_m)\| = O(t^{1/p_1}) \ \text{as } t \to 0. \tag{12.25}$$

Proof. Estimate (12.24) for $t \in \overline{\mathbb{R}}_+$ was obtained in Corollary 12.7.1. This property implies that assertions (Step 1) and (Step 2) hold. Therefore, as in Sec. 12.1, we obtain that A_m generates a C_0-semigroup $\exp(tA_m)$, $t \in \overline{\mathbb{R}}_+$.

On the other hand, we fix $T > 0$ and set $M = \sup\limits_{-T \le t \le T} \|\exp(tA_m)\|$. Identity (12.22), together with (12.24), yields the following for $0 \le t \le T$:

$$\|\exp(tA_m) - \exp(tA)\| \le \|K_t\|_{B(L^\infty([0,T];E),L^\infty([0,T];E))} \le C L e^{\alpha T} |T|^{1/p_1}, \tag{12.26}$$

from which we obtain (12.25) for $t \in \mathbb{R}$.

It remains to show that A_m generates a C_0-group and (12.26) also holds for $t < 0$. Choose $0 < T_0 \le T$ so that $M^2 L e^{\alpha T_0} T_0^{1/p_1} \le 1/2$. Take $0 \le t \le T_0$. Then

$$\exp(tA_m) = \Big(\big(\exp(tA_m) - \exp(tA)\big)\exp(-tA) + I\Big)\exp(tA).$$

Further, by (12.26), for $t > 0$, and by the choice of T_0, we also have

$$\Big\|\Big(\exp(tA_m) - \exp(tA)\Big)\exp(-tA)\Big\| \le 1/2.$$

Therefore, the series expansion shows that the operator $\exp(\cdot A_m)$ is invertible and $\|\exp(tA_m)^{-1}\| \le 2M$. In fact, this proves that A_m generates a C_0-group.

Finally, using the series expansion once again, we obtain $0 \le t \le T_0$,

$$\|\exp(-tA_m) - \exp(-tA)\| \le \sum_{j=1}^{+\infty} \|\Big(\exp(tA_m) - \exp(tA)\Big)\exp(-tA)\|^j$$
$$= \sum_{j=1}^{+\infty} \left(M^2 L e^{\alpha t} t^{1/p_1}\right)^j \le 2M^2 L e^{\alpha t} t^{1/p_1},$$

which proves (12.26) also for $t < 0$.

For Hilbert spaces the value $p_1 = +\infty$ is also admissible. Indeed, using Corollary 12.7.2, we obtain the following theorem.

Theorem 12.9.2 ([232]). *Assume that $E = H$ is Hilbert and there exist $T_0 > 0$ and $p_1, p_2 \in [1, +\infty]$, such that $K_T \in B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))$. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/2}, \qquad T \in \mathbb{R}. \tag{12.27}$$

Moreover, the operator A_m generates a C_0-group and

$$\|\exp(tA) - \exp(tA_m)\| = O(t^{1/2}) \text{ as } t \to 0.$$

Finally, under Condition HG, we can apply Theorem 12.7.2, which leads to the following theorem, which is true for Banach spaces.

Theorem 12.9.3 ([232]). *Let Condition* HG *hold. Also, assume that there exist $T_0 > 0$ and $p_1, p_2 \in [1, +\infty]$, such that $\|K_{T_0}\|_{B(L^{p_1}([0,T_0],E), L^{p_2}([0,T_0],E))} < +\infty$. Then $A_m = A(I + B)$ is a generator of a C_0-group, and we have the estimate*

$$\|\exp(tA) - \exp(tA_m)\| = O(t) \text{ as } t \to 0.$$

Moreover, if E is reflexive, then B takes its values in the domain of A, AB is a bounded operator, and $A_m = A + AB$ is a bounded perturbation of A.

12.10. Perturbations of C_0-Cosine Operator Functions

Let $C(\cdot, A)$ be a C_0-cosine family of linear bounded operators on a Banach space E. Taking a linear bounded operator B on E, we pose the problem: whether the multiplicatively perturbed operator $A_m = A(I + B)$, acting on the domain $D(A_m) = \{x \in E : (I + B)x \in D(A)\}$ is a generator of another

C_0-cosine family or not. First, assume that A_m is a generator of a C_0-cosine family $C(\cdot, A_m)$. We have already known that

$$C(t, A_m)x = C(t, A)x + A\int_0^t S(t-s, A)BC(s, A_m)x\,ds, \qquad t \in \mathbb{R}, \quad x \in E. \tag{12.28}$$

Acting in the same way as in Sec. 12.9, for a fixed $x \in E$, we arrive at the convolution equation

$$f(t) = C(t, A)x + (K_T f)(t), \qquad 0 \le t \le T, \tag{12.29}$$

where K_T is defined in (12.19) and f is sought in $C([0,T]; E)$. As was shown in Sec. 12.1, if

(i) there exists $T > 0$ such that K_T continuously maps $C([0,T]; E)$ onto itself and

(ii) $\mathrm{SV}(K_T, T) = \|K_T\|_{B(L^\infty([0,T];E), L^\infty([0,T];E))} < 1$ for a sufficiently small $T > 0$,

then by the contraction mapping principle, Eq. (12.29) is uniquely solvable and A_m is a generator of a C_0-cosine family.

Theorem 12.10.1 ([232]). *Assume that there exist $T_0 > 0$, $1 \le p_1 < +\infty$, and $1 \le p_2 \le +\infty$ such that $\|K_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < \infty$. Then there exist $L > 0$ and $\alpha > 0$ such that for any $T > 0$, the following inequality holds*:

$$\mathrm{SV}(K_T, T) \le Le^{\alpha T}T^{1/p_1}. \tag{12.30}$$

Moreover, the operator A_m generates a C_0-cosine operator family, and

$$\|C(t, A) - C(t, A_m)\| = O(t^{1/p_1}) \quad \text{as } t \to 0. \tag{12.31}$$

Using Corollary 12.8.3, we easily obtain the following theorem.

Theorem 12.10.2 ([232]). *Assume that $E = H$ is a Hilbert space and that there exist $T_0 > 0$ and $p_1, p_2 \in [1, +\infty]$ such that $\|K_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < +\infty$. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le Le^{\alpha T}T^{1/2}, \qquad T > 0. \tag{12.32}$$

Moreover, the operator A_m generates a C_0-cosine operator family, and

$$\|C(t, A) - C(t, A_m)\| = O(t^{1/2}) \quad \text{as } t \to 0.$$

Theorem 12.10.3 ([232]). *Assume that Condition* HC *holds. Also, assume that there exist $T_0 > 0$ and $p_1, p_2 \in [1, +\infty]$ such that $\|K_{T_0}\|_{B(L^{p_1}([0,T_0];E), L^{p_2}([0,T_0];E))} < +\infty$. Then $A_m = A(I + B)$ is a generator of a C_0-cosine operator family, and the following estimate holds*:

$$\|C(t, A) - C(t, A_m)\| = O(t) \quad \text{as } t \to 0.$$

INHOMOGENEOUS EQUATIONS

In a Banach space E, let us consider the inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t), \quad t \in [0, T], \quad u(0) = u^0, \tag{13.1}$$

with the operator A generating a C_0-semigroup. For $f \equiv 0$, the well-posed statement of such problems are described in detail in Chapter 1 of [17]. Formally, as in the finite-dimensional analysis, problem (13.1) has a solution of the form

$$u(t) = \exp(tA)u^0 + \int_0^t \exp((t-s)A)f(s)\, ds, \quad t \in [0, T]. \tag{13.2}$$

This is the so-called constant variation formula. The properties of the expression $\int\limits_0^t \exp((t-s)A)f(s)\, ds$ and the corresponding interpretations of solutions related to representation (13.2) are of interest.

13.1. General Results

It is known from the theory of partial differential equations that problems that are written in abstract form (13.1) are usually considered in the spaces of types $C([0, T]; E)$ or $L^p([0, T]; E)$. In this chapter, we restrict ourselves to these two statements.

Thus, for example, if $f(\cdot) \in C([0, T]; E)$, then for the C_0-semigroup $\exp(\cdot A)$, the expression $\exp((t - s)A)f(s)$ is continuous in $s \in [0, T]$, and hence there exists $\int\limits_0^t \exp((t-s)A)f(s)\, ds$.

On the other hand, if $u(\cdot)$ is a solution of problem (13.1) with $f(\cdot) \in C([0, T]; E)$, which belongs to $C([0, T]; E) \cap C^1((0, T]; E)$, then $\dfrac{d}{ds}\Big(\exp((t-s)A)u(s)\Big) = \exp((t-s)A)f(s)$ and integrating over $(0, t)$, we obtain (13.2). The converse is not true in general, since the function $u(\cdot)$ given by expression (13.2) can be not differentiable.

In the theory of abstract differential equation, the following theorem is well known.

Theorem 13.1.1 ([20, 31]). *Let $A \in \mathcal{G}(M, \omega)$, $u^0 \in D(A)$, and let the function $f(\cdot)$ be such that either*

(i) $f(\cdot) \in C^1([0, T]; E)$

or

(ii) $f(\cdot) \in C([0, T]; E)$ *takes its values in* $D(A)$ *and, moreover,* $Af(\cdot) \in C([0, T]; E)$.

Then problem (13.1) *has a unique solution* $u(\cdot) \in C^1([0, T]; E)$ *with the initial condition* u^0 *that is represented in form* (13.2).

In the case where $f(\cdot)$ satisfies the Hölder condition

$$\|f(t) - f(s)\| \leq M|t - s|^\gamma \quad \text{for} \quad 0 \leq s, t \leq T, \tag{13.3}$$

with certain constants $M > 0$ and $0 < \gamma \leq 1$, the following theorem holds.

Theorem 13.1.2 ([20, 31]). *Let $A \in \mathcal{H}(\omega, \theta)$, and let $f(\cdot)$ satisfy the Hölder condition* (13.3). *Then the function $u(\cdot)$ from* (13.2) *belongs to $C([0,T]; E) \cap C^1((0,T]; E)$ and is a solution of problem* (13.1) *for any $u^0 \in E$. Moreover, $u(\cdot) \in C^1([0,T]; E)$ if $u^0 \in D(A)$.*

The Cauchy–Kowalewski theorem can also be written in an abstract form. Let $\{E_\theta : 0 \leq \theta \leq 1\}$ be Banach spaces having the properties $E_{\theta_2} \subseteq E_{\theta_1}$ if $\theta_1 < \theta_2$ and

$$\|x\|_{E_{\theta_1}} \leq \|x\|_{E_{\theta_2}} \quad \text{for any} \quad x \in E_{\theta_2}.$$

Let L_α be the set of linear operators $Q \in L(E_{\theta_2}, E_{\theta_1})$ for $0 \leq \theta_1 < \theta_2 < 1$ such that

$$\|Qx\|_{E_{\theta_1}} \leq \frac{\alpha}{\theta_2 - \theta_1} \|x\|_{E_{\theta_2}} \quad \text{for any} \quad x \in E_{\theta_2}.$$

A function $A(t) : [0,T] \to L_\alpha$ is said to be continuous if for any $\varepsilon > 0$, $t_0 \in [0,T]$, and $\theta > \theta'$, there exists $\delta > 0$ such that for $|t - t_0| < \delta$, we have

$$\|(A(t) - A(t_0))x\|_{E_{\theta'}} \leq \varepsilon \|x\|_{E_\theta} \quad \text{for any } x \in E_\theta.$$

We call attention to the fact that the space L_α is a Banach space with the norm

$$\|Q\|_{L_\alpha} = \sup_{0 \leq \theta' < \theta \leq 1} \sup_x (\theta - \theta') \|Qx\|_{E_{\theta'}} \|x\|_{E_\theta}^{-1}.$$

Theorem 13.1.3 ([29]). *Let $u^0 \in E_1$, $f(\cdot) \in C([-T,T]; E_1)$, and let $A(\cdot) \in C([-T,T]; L_\alpha)$. Then*

1. *For each $\theta \in [0,1)$, there exists a function $u(\cdot)$ defined for $0 \leq t < T_s := \min(T, (1-\theta)(\alpha e)^{-1})$ and taking its values in E_θ. The function $u(\cdot)$ is continuously differentiable and satisfies the equation*

$$u'(t) = A(t)u(t) + f(t) \quad \textit{for} \quad 0 \leq t < T_s \tag{13.4}$$

and the condition $u(0) = u^0$.

2. *If for certain $\theta \in (0,1]$ and $0 < T' \leq T$, on the set $[0,T']$, we have two functions with values in E_θ that are continuously differentiable, satisfy* (13.4), *and coincide for $t = 0$, then these functions coincide on $[0,T']$.*

For a second order equation, in a formula of form (13.2), instead of $\exp(\cdot A)$ we have $C(\cdot, A)$ and $S(\cdot, A)$. However, often, there are no technical distinctions in studying these problems, and, as a rule, we restrict ourselves to proofs only for the case of second order equations.

13.2. Inhomogeneous Equations in $C([0,T];E)$

In a Banach space E, let us consider problem (13.1) with the operator A generating a C_0-semigroup.

Definition 13.2.1. A *classical solution* of problem (13.1) is a function $u(\cdot)$ such that $u(\cdot) \in C^1([0,T];E)$, $u(t) \in D(A)$ for all $t \in [0,T]$, and relations (13.1) hold.

Proposition 13.2.1 ([235]). *Let $f(\cdot) \in L^1([0,T];E)$. Then for any $u^0 \in E$, problem* (13.1) *has not more than one classical solution. If it has a classical solution, then this solution has the form* (13.2).

Definition 13.2.2. A *weakened solution* of problem (13.1) is a function $u(\cdot) \in C([0,T];E)$ such that $u'(\cdot) \in C((0,T];E)$ and Eq. (13.1) holds on $(0,T]$.

Theorem 13.2.1 ([47]). *Let problem* (13.1) *with $f(\cdot) \in C([0,T];E)$ and $u^0 \in D(A)$ have a weakened solution $u(\cdot)$, and let $\rho(A) \neq \varnothing$. Then $u(\cdot)$ is given in the form* (13.2).

As was noted, the function $u(\cdot)$ given in the form (13.2) is neither a classical nor a weakened solution in general, since it can be not differentiable.

Definition 13.2.3. A function $u(\cdot) \in C([0,T];E)$ given by (13.2) is called a *mild solution* of problem (13.1).

Theorem 13.2.2 ([235]). *Let $A \in \mathcal{G}(M,-\omega)$ with $\omega > 0$, and let the function $f(\cdot) : [0,\infty) \to E$ be bounded and measurable on $[0,\infty)$. If $s\text{-}\lim\limits_{t\to\infty} f(t) = f_\infty$, then a mild solution $u(\cdot)$ defined by* (13.2) *has the following behavior:*

$$s\text{-}\lim_{t\to\infty} u(t) = -A^{-1} f_\infty.$$

In a Banach space E, let us consider the Cauchy problem

$$u''(t) = Au(t) + f(t), \quad t \in [0,T], \quad u(0) = u^0, \ \ u'(0) = u^1, \tag{13.5}$$

with the operator A generating a C_0-cosine operator function.

Definition 13.2.4. A function $u(\cdot)$ is called a *classical solution* of problem (13.5) if $u(\cdot)$ is twice continuously differentiable, $u(t) \in D(A)$ for all $t \in [0,T]$, and $u(\cdot)$ satisfies relations (13.5).

If $f(\cdot) \in C([0,T];E)$ and $u(\cdot)$ is a classical solution of (13.5), then, considering the expression $\dfrac{d}{ds}\Big(C(t-s,A)u(s) + S(t-s,A)u'(s)\Big) = S(t-s,A)f(s)$ and integrating it in $0 \le s < t$, we obtain

$$u(t) = C(t,A)u^0 + S(t,A)u^1 + \int_0^t S(t-s,A)f(s)\,ds, \quad t \in [0,T]. \tag{13.6}$$

As in the case of C_0-semigroups of operators, the function $u(\cdot)$ given by (13.6) is not a classical solution in general, since it can be not twice continuously differentiable.

Proposition 13.2.2 ([135]). *Let $A \in \mathcal{C}(M, \omega)$, and let either*

(i) $f(\cdot), Af(\cdot) \in C([0,T); E)$ *and* $f(t) \in D(A)$ *for* $t \in [0, T]$

or

(ii) $f(\cdot) \in C^1([0,T]; E)$.

Then the function $u(\cdot)$ from (13.6) *with $u^0 \in D(A)$ and $u^1 \in E^1$ is a classical solution of problem* (13.5) *on* $[0, T]$.

Definition 13.2.5. The function $u(\cdot) \in C([0,T); E)$ given by expression (13.6) is called a *mild solution* of problem (13.5).

Theorem 13.2.3 ([47]). *Let the operator $B = \sqrt{A}$ in problem* (13.5) *have a bounded inverse $B^{-1} \in B(E)$ and be a generator of a C_0-group, and let the function $f(\cdot)$ have one of the following properties*:

(i) $f(\cdot) \in C^1([0,T); E)$;

(ii) $Bf(\cdot) \in C([0,T); E)$.

Then for any $u^0 \in D(A)$ and $u^1 \in D(B)$, there exists a unique classical solution of problem (13.5) *given by formula* (13.6) *in the form*

$$\begin{aligned} u(t) = {} & \frac{1}{2}\Big(\exp(tB) + \exp(-tB)\Big)u^0 + \frac{1}{2}\Big(\exp(tB) - \exp(-tB)\Big)B^{-1}u^1 \\ & + \frac{1}{2}\int_0^t \Big(\exp((t-s)B) + \exp(-(t-s)B)\Big)B^{-1}f(s)\,ds, \quad t \in [0,T]. \end{aligned} \tag{13.7}$$

Theorem 13.2.4 ([106]). *Let $A \in \mathcal{C}(M, \omega)$. The Cauchy problem*

$$u''(t) = Au(t) + f(t), \quad u(0) = x, \quad u'(0) = y, \tag{13.8}$$

has a unique 2π-periodic mild solution of class C^1 for any 2π-periodic function $f(\cdot) \in L^2_{\mathrm{loc}}(\mathbb{R}; E)$ iff $1 \in \rho\Big(C(2\pi, A)\Big)$.

Note that the condition $1 \in \rho(C(\pi, A))$ is equivalent to the conditions $\{-4\pi^2k^2/T^2\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\sup\limits_{k\in\mathbb{Z}} \|k(4\pi^2k^2/T^2 + A)^{-1}\| < \infty$.

13.3. Coercivity in the Case of Classical Solutions

Since the existence of a classical solution of problem (13.1) presupposes the continuity of the derivative of the function $u(\cdot)$ on $[0, T]$, then by representation (13.2), it is natural to consider the differentiability of the expression $(\exp(\cdot A) * f)(t)$ in the variable $t \in \mathbb{R}_+$.

Definition 13.3.1. We say that a C_0-semigroup $\exp(\cdot A)$ has the property of *maximal regularity* (MR-property in brief) if

$$(\exp(\cdot A) * f)(\cdot) \in C^1([0,T]; E)$$

$$\text{(or, which is equivalent, } (\exp(\cdot A) * f)(\cdot) \in C([0,T]; \mathcal{D}(A)) \text{ for all } f(\cdot) \in C([0,T]; E)).$$

Proposition 13.3.1 ([124]). *The convolution* $(\exp(\cdot A) * f)(\cdot) \in C^1([0,T]; E)$ *iff* $(\exp(\cdot A) * f)(t) \in D(A)$ *for all* $t \in [0,T]$ *and* $(\exp(\cdot A) * f) \in C([0,T]; \mathcal{D}(A))$.

Proposition 13.3.2 ([124]). *The Cauchy problem* (13.1) *has a classical solution for any* $f \in C([0,T]; E)$ *iff* A *generates a* C_0*-semigroup with the MR-property.*

Proposition 13.3.3 ([271]). *An operator* A *generates a* C_0*-semigroup with the MR-property iff* $\mathrm{SV}(\exp(\cdot A), t)$ *is bounded on* $[0,T]$.

Theorem 13.3.1 ([89, 124]). *If a* C_0*-semigroup* $\exp(\cdot A)$ *has the MR-property, then either* A *is bounded or the space* E *contains a closed subspace isomorphic to* c_0.

As was shown in [124], there exist unbounded operators on $E = c_0$ that generate C_0-semigroups with the MR-property.

Since an operator A generating a C_0-semigroup is closed, by the closed graph theorem, in the case of a C_0-semigroup with the MR-property, the operator $A(\exp(\cdot A) * f)$ defined on the whole $C([0,T]; E)$ is continuous as an operator from $C([0,T]; E)$ into $C([0,T]; E)$. This means that the following inequality holds:

$$\|A(\exp(\cdot A) * f)\|_{C([0,T];E)} \le C\|f\|_{C([0,T];E)}$$

with a certain constant C independent of $f(\cdot)$. Generalizing the clearness of the previous inequality for formulating the MR-property in the space $C([0,T]; E)$ to the description of well-posedness of the Cauchy problem for an inhomogeneous equation in spaces of types C^α, h^α, and so on, we arrive at the following definition.

Definition 13.3.2. Let F be a Banach space being a subspace of the initial space E, $\Upsilon([0,T]; E)$ be the Banach space of functions with values in E. Problem (13.1) is said to be *coercively solvable* in the pair of spaces $(F, \Upsilon([0,T]; E))$ (i.e., the solution $u(\cdot)$ has the *maximal regularity property*) if for any $u^0 \in F$ and any right-hand side $f(\cdot) \in \Upsilon([0,T]; E)$, there exists a classical solution $u(\cdot)$ of the Cauchy problem (13.1), and for this solution, we have the coercive inequality

$$\|u'(\cdot)\|_{\Upsilon([0,T];E)} + \|Au(\cdot)\|_{\Upsilon([0,T];E)} \le M(\|f(\cdot)\|_{\Upsilon([0,T];E)} + \|u_0\|_F). \tag{13.9}$$

The formulation of this definition is very convenient. So, for example, Definition 13.3.2 for $u(t) = t\exp(tA)u^0$ and $f(t) = \exp(tA)u^0$ and Proposition 13.3.3 trivially imply the following assertion.

Proposition 13.3.4 ([271]). *Let A be a generator of a C_0-semigroup, and let $\mathrm{SV}(\exp(\cdot A), t) < \infty$. Then the semigroup $\exp(\cdot A)$ is analytic.*

However, the analyticity of the C_0-semigroup $\exp(\cdot A)$ is not sufficient for the coercive solvability of problem (13.1) in $C([0,T];E)$ (see [72]). Therefore, taking into account Theorem 13.3.1, we see that the study of coercivity in the space $C([0,T];E)$ is not interesting. However, one can prove the following theorem.

Theorem 13.3.2 ([219]). *Let $-\infty < a < b < \infty, 1 \le p, q \le \infty, \sigma \ge 0$, and let $A \in \mathcal{H}(\theta, \beta)$. Then for any $f \in B^{\sigma}_{p,q}((a,b);E)$ with $\sigma > 1/p$ or $\sigma = 1/p$ and $q = 1$, we have $\exp(\cdot A)*f \in C^1((a,b);E) \cap C^1((a,b);\mathcal{D}(A))$ and (13.1) holds for any $t \in (a,b)$ for the function $u(\cdot) = \exp(\cdot A) * f$.*

In fact, under the condition of the previous theorem, we can prove that for any $f \in B^{\sigma}_{p,q}((a,b);E) \cap L^1((a,b);E)$, it follows that $\exp(\cdot A) * f \in B^{\sigma+1}_{p,q}((a,b);E)$, where $a < a_1 < b$, and there exists a constant $c > 0$ with the property

$$\| \exp(\cdot A) * f\|_{B^{\sigma+1}_{p,q}((a,b);E)} \le c(\|f\|_{B^{\sigma}_{p,q}((a,b);E)} + \|f\|_{L^1((a,b);E)}).$$

However, such estimates are not coercive.

Proposition 13.3.5 ([70]). *Let $A \in \mathcal{H}(\omega, \theta)$, i.e., A generates a certain analytic C_0-semigroup, $F = \mathcal{D}(A)$, and $\Upsilon([0,T];E))$ is the Hölder space of functions $C_0^{\alpha}([0,T];E)$ for which the following norm is finite:*

$$\|f(\cdot)\|_{C_0^{\alpha}([0,T];E)} = \sup_{0\le t\le T} \|f(t)\|_E + \sup_{0\le t,\tau,t+\tau\le T} \frac{\|f(t+\tau) - f(t)\|_E}{\tau^{\alpha}} t^{\alpha}.$$

Then the Cauchy problem (13.1) is coercively solvable in the pair of spaces $(F, C_0^{\alpha}([0,T];E))$.

Theorem 13.3.3 ([3]). *Let $v' = f(0) - Au(0) \in E_{\alpha-\gamma}$, $f \in C_0^{\beta,\gamma}([0,1];E_{\alpha-\gamma})$, $A \in \mathcal{H}(\omega,\theta)$ for $0 \le \gamma \le \beta \le \alpha$, $0 < \alpha < 1$. Then there exists a unique solution of problem (13.1), $Au, u' \in C_0^{\beta,\gamma}([0,1];E_{\alpha-\gamma})$, $u' \in C([0,1];E_{\alpha-\gamma})$, and*

$$\|u'\|_{C_0^{\beta,\gamma}(E_{\alpha-\gamma})} + \|Au\|_{C_0^{\beta,\gamma}(E_{\alpha-\gamma})} + \|u'\|_{C([0,1];E_{\alpha-\gamma})} \le c\left(\|v_0'\|_{\alpha-\gamma} + \frac{1}{\alpha(1-\alpha)}\|f\|_{C_0^{\beta,\gamma}([0,1];E_{\alpha-\gamma})}\right),$$

where $C_0^{\beta,\gamma}$ has the norm

$$\max_t \|f\|_E + \max_{0<t,t+\tau\le T} \frac{\|f(t+\tau) - f(t)\|_E}{\tau^{\beta}} (t+\tau)^{\gamma}.$$

Remark 13.3.1 ([68]). In the case where $E = l^p$, the operator

$$A(x_k)_{k=1}^{\infty} = (ik\, x_k)_{k=1}^{\infty} \quad (i = \sqrt{-1}),$$

generates a strongly continuous C_0-semigroup (but not an analytic one!). For the right-hand side $f(t) = \big(k^{-(1-1/p)} e^{ikt}\big)_{k=1}^{\infty}$ satisfying the Hölder condition with any exponent $\varepsilon \in (0,1)$, the function

$$\phi(t) = \int_0^t \exp((t-s)A) f(s)\, ds$$

does not belong to $D(A)$ for any t.

This example also shows that in the case where the operator A generates a C_0-semigroup only, the Hölder property of the right-hand side is not sufficient for the existence of a classical solution.

Denote $(S(\cdot, A) * f)(t) := \int\limits_0^t S(t-s, A) f(s)\, ds$, $t \in [0,T]$.

Definition 13.3.3. We say that a C_0-cosine operator function $C(\cdot, A)$ has the *maximal regularity* (*MR-property*) if $S(\cdot, A) * f \in C^2([0,T]; E)$ (or, which is equivalent, $C(\cdot, A) * f \in C([0,T]; \mathcal{D}(A))$ for all $f(\cdot) \in C([0,T]; E)$.

Proposition 13.3.6 ([100]). *Let $x, y \in D(A)$. Then the following assertions are equivalent*:

(i) *problem* (13.5) *has a classical solution for a given* $f(\cdot)$;

(ii) $S(\cdot, A) * f \in C^2([0,T]; E)$;

(iii) $(S(\cdot, A) * f)(t) \in D(A)$ *for* $0 \le t \le T$, *and* $A(S(\cdot, A) * f)(t)$ *is continuous in* $t \in [0,T]$, *i.e.*, $(S(\cdot, A) * f) \in C([0,T], \mathcal{D}(A))$.

Proof. (i) $\Longrightarrow$ (ii). We know that if $u(\cdot)$ is a solution of (13.5), then $u(\cdot)$ is twice continuously differentiable and $u(t) = C(t,A)x + S(t,A)y + (S(\cdot,A) * f)(t)$. Therefore, we have $(S(\cdot, A) * f)''(t) = u''(t) - C''(t,A)x - S''(t,A)y = u''(t) - C(t,A)Ax - S(t,A)Ay \in C([0,T]; E)$, i.e., $S(\cdot, A) * f \in C^2([0,T]; E)$.

(ii) $\Longrightarrow$ (iii). Since

$$\begin{aligned}
\frac{2}{h^2}(C(h,A) - I)(S(\cdot,A) * f)(t) &= \frac{1}{h^2}((S(\cdot,A) * f)(t+h) - 2(S(\cdot,A) * f)(t) + (S(\cdot,A) * f)(t-h)) \\
&+ \frac{1}{h^2}\left(-\int_t^{t+h} S(t-s+h,A) f(s)\, ds + \int_{t-h}^t S(t-s-h,A) f(s) ds\right) = \\
&= \frac{1}{h^2}\Big((S(\cdot,A) * f)(t+h) - 2(S(\cdot,A) * f)(t) + (S(\cdot,A) * f)(t-h)\Big) \\
&+ \frac{1}{h^2}\left(-\int_t^{t+h} S(t-s+h,A) f(s) ds + \int_{t-h}^t S(t-s-h,A) f(s)\, ds\right), \qquad (13.10)
\end{aligned}$$

we have

$$\lim_{h \to 0+} \frac{2}{h^2}(C(t,A) - I)(S(\cdot,A) * f)(t) = (S(\cdot,A) * f)''(t) - f(t),$$

i.e., $(S(\cdot,A)*f)(t) \in D(A)$ and $A(S(\cdot,A)*f)(t) = (S(\cdot,A)*f)''(t) - f(t)$. Therefore, $A(S(\cdot,A)*f)(\cdot) \in C([0,T];E)$.

(iii) $\Longrightarrow$ (i). By (13.10),

$$\frac{1}{h^2}\Big((S(\cdot,A)*f)(t+h) - 2(S(\cdot,A)*f)(t) + (S(\cdot,A)*f)(t-h)\Big) = \frac{2}{h^2}\Big(C(h,A) - I\Big)(S(\cdot,A)*f)(t)$$
$$-\frac{1}{h^2}\Big(- \int_t^{t+h} S(t-s+h,A)f(s)ds \quad + \int_{t-h}^{t} S(t-s-h,A)f(s)ds\Big).$$

Then (c) implies $(S(\cdot,A)*f)'' = A(S(\cdot,A)*f) + f \in C([0,T];E)$. Therefore, $S(\cdot,A)*f$ is a solution of the Cauchy problem (13.5) for a given $f(\cdot)$ and zero initial data, and $u(t) = C(t,A)x + S(t,A)y + (S*f)(t)$, $t \in \mathbb{R}$, is a solution of the Cauchy problem (13.5) for a given $f(\cdot)$ for each pair $x, y \in D(A)$.

Theorem 13.3.4 ([100]). *For a cosine operator function $C(\cdot,A)$, the following conditions are equivalent*:

(i) *the generator A is bounded*;

(ii) $\|C(t,A) - I\| = O(t^2)$ $(t \to 0^+)$;

(iii) $\mathrm{Var}(C(\cdot,A),t) = O(t^2)\,(t \to 0^+)$;

(iv) $\mathrm{Var}(C(\cdot,A),t) = o(1)\,(t \to 0^+)$;

(v) $\mathrm{SV}(C(\cdot,A),t) = O(t^2)\,(t \to 0^+)$;

(vi) $\mathrm{SV}(C(\cdot,A),t) = o(1)\,(t \to 0^+)$;

(vii) $\mathrm{SV}(C(\cdot,A),t) < \infty$ *for some $t > 0$, i.e., $C(\cdot,A)$ is locally of bounded semivariation*;

(viii) *$R(S(t,A)) \subseteq D(A)$ for $t \in (-\infty,\infty)$, and $\|AS(t,A)\|$ is bounded on $[a,b]$ for some $0 < a < b$*;

(ix) *$R(S(t,A)) \subseteq D(A)$ for $t \in (-\infty,\infty)$ and $\limsup\limits_{t\to 0^+} \|tAS(t,A)\| < \dfrac{2}{e}$.*

Definition 13.3.4. Let F be a Banach space being a subspace of the initial space E, and let $\Upsilon([0,T];E)$ be the Banach space of functions with values in E. Problem (13.5) is said to be *coercively solvable* in the pair of spaces $(F, \Upsilon([0,T];E))$ (in other words, the solution has $u(\cdot)$ the *maximal regularity property*) if for any right-hand side $f(\cdot) \in \Upsilon([0,T];E)$, there exists a classical solution $u(\cdot)$ of the Cauchy problem (13.5), for each t, the value of the solution $u(t)$ belongs to F, and the following coercive inequality holds for it:

$$\|u''(\cdot)\|_{\Upsilon([0,T];E)}\| + \|Au(\cdot)\|_{\Upsilon([0,T];E)} \le M\Big(\|f(\cdot)\|_{\Upsilon([0,T];E)} + \|u^0\|_F + \|u^1\|_F\Big).$$

Theorem 13.3.5 ([100]). *Let problem* (13.5) *be coercively solvable in the pair* $(\mathcal{D}(A), C([0,T];E))$. *Then* $A \in B(E)$.

This result can also be reformulated as follows:

Theorem 13.3.6 ([100]). *The following statements are equivalent:*

(i) *for all $x, y \in D(A)$ and $f \in C([0,r];E)$, problem* (13.5) *has a classical solution;*

(ii) *the operator A generates a C_0-cosine operator function that satisfies the MR-property;*

(iii) *the operator A generates a C_0-cosine operator function that is of bounded semivariation on* $[0,r]$;

(iv) *A is a bounded linear operator on E.*

13.4. Coercivity in $L^p([0,T];E)$

Let $1 \le p \le \infty$. Then [92] $u(\cdot) \in W^{1,p}([0,T];E)$ iff $u(\cdot)$ is absolutely continuous and $u(\cdot), u'(\cdot) \in L^p([0,T];E)$.

Definition 13.4.1. A function $u(\cdot)$ is said to be *absolutely continuous* if there exists a function $v(\cdot) \in L^1([0,T];E)$ such that

$$u(t) = u(\tau) + \int_\tau^t v(\xi)\,d\xi \quad \text{for all} \quad \tau, t \in [0,T].$$

An absolutely continuous function $u(\cdot)$ is continuous and differentiable almost everywhere on $[0,T]$, and, moreover, $u'(\cdot) = v(\cdot)$.

Definition 13.4.2. A *classical solution* of problem (13.1) in the space $L^p([0,T];E)$ is an absolutely continuous function $u(\cdot)$ such that the function $u'(\cdot), Au(\cdot) \in L^p([0,T];E)$ satisfies Eq. (13.1) almost everywhere on $[0,T]$ and $u(0) = u^0$.

Definition 13.4.3. Problem (13.1) is said to be *well-posed in* $L^p([0,T];E)$ if for any $f(\cdot) \in L^p([0,T];E)$, there exists a unique classical solution $u(\cdot)$ in $L^p([0,T];E)$ continuously depending on u^0 and $f(\cdot)$.

Definition 13.4.4. Problem (13.1) is said to be *coercively solvable in* $L^p([0,T];E)$ if for any $f(\cdot) \in L^p([0,T];E)$ and $u^0 \in F$, there exists a unique classical solution of problem (13.1) in $L^p([0,T];E)$ and

$$\|u'(\cdot)\|_{L^p([0,T];E)} + \|Au(\cdot)\|_{L^p([0,T];E)} \le M(p)(\|f(\cdot)\|_{L^p([0,T];E)} + \|u^0\|_F). \tag{13.11}$$

Recall that if a certain property holds locally, then we write L^p_{loc} instead of L^p.

Definition 13.4.5. A function $u(\cdot)$ is called a $W^{1,p}_{\text{loc}}$*-solution* of the Cauchy problem (13.1) if $u(\cdot) \in W^{1,p}_{\text{loc}}([0,T];E) \cap L^p([0,T];\mathcal{D}(A))$ and (13.1) is satisfied.

Proposition 13.4.1 ([77]). *Let $1 \le p \le \infty$, $A \in \mathcal{H}(\theta,\omega)$, and let $u(\cdot)$ is a $W^{1,p}_{\text{loc}}$-solution of the Cauchy problem* (13.1). *Then we have representation* (13.2) *for this solution.*

Proposition 13.4.2 ([72]). *The coercive solvability of problem* (13.1) *in* $L^p([0,T];E)$ *and the compactness of the resolvent $(\lambda I - A)^{-1}$ imply the analyticity of the C_0-semigroup* $\exp(\cdot A)$.

To understand which space F should be in Definition 13.4.3, we consider the function $A\exp(tA)u^0$. Obviously, F is the space with the norm

$$\|u^0\|_F = \left(\int_0^T \|A\exp(tA)u^0\|_E^p\,dt\right)^{1/p} + \|u^0\|_E. \tag{13.12}$$

Such a space F is a particular case of the spaces $E_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, with the norm

$$\|u^0\|_{E_{\theta,p}} = \|u^0\|_E + \left(\int_0^T \|t^{1-\theta}A\exp(tA)u^0\|_E^p\,\frac{dt}{t}\right)^{1/p},$$

$$\|u^0\|_{E_{\theta,\infty}} = \sup_{\tau>0} \tau^{1-\theta}\|A\exp(\tau A)u^0\|_E.$$

We denote it by $E_{1-\frac{1}{p}} = E_{1-\frac{1}{p},p}$.

Theorem 13.4.1 ([82]). *For any function $f(\cdot) \in L^p([0,T];E)$ and any $u^0 \in E_{1-\frac{1}{p}}$, formula (13.2) defines an $E_{1-\frac{1}{p}}$-valued function $u(\cdot)$ on $[0,T]$ and*

$$\max_{0\le t\le T} \|u(t)\|_{E_{1-\frac{1}{p}}} \le M\left(\|u^0\|_{E_{1-\frac{1}{p}}} + \frac{p^2}{p-1}\|f\|_{L^p([0,T];E)}\right).$$

Theorem 13.4.2 ([82]). *Let problem (13.1) be coercively solvable in the space $L^{p_0}([0,T];E)$ for a certain $1 < p_0 < \infty$ with $M(p_0) = M$. Then it is coercively solvable for any $1 < p < \infty$, and estimate (13.11) holds with $M(p) = M\dfrac{p^2}{p-1}$.*

Theorem 13.4.3. *If the space $E = H$ is Hilbert and $A \in \mathcal{H}(\omega,\theta)$, then problem (13.1) is coercively solvable in $L^2([0,T];H)$.*

Theorem 13.4.4 ([82]). *Under the conditions of Theorem 13.4.2, for any $f(\cdot) \in L^p([0,T];E)$ and $u^0 \in E_{1-\frac{1}{p}}$, problem (13.1) has a unique solution $u(\cdot)$ in $L^p([0,T];E)$ such that the coercive inequality (13.11) holds in the form*

$$\|u'\|_{L^p([0,T];E)} + \|Au\|_{L^p([0,T];E)} + \max_{0\le t\le T}\|u(t)\|_{E_{1-\frac{1}{p}}} \le M\frac{p^2}{p-1}\left(\|f\|_{L^p([0,T];E)} + \|u^0\|_{E_{1-\frac{1}{p}}}\right).$$

Theorem 13.4.5 ([82]). *Let $f(\cdot) \in L^q([0,T];E_{\alpha,q})$, $0 < \alpha < 1$, $1 \le q \le \infty$, and let $A \in \mathcal{H}(\theta,\omega)$. Then there exists a unique absolutely continuous solution $u(\cdot)$ of problem (13.1) with $u^0 = 0$ such that $Au(\cdot), u'(\cdot) \in L^q([0,T];E_{\alpha,q})$, and*

$$\|u'\|_{L^q([0,T];E_{\alpha,q})} + \|Au\|_{L^q([0,T];E_{\alpha,q})} \le \frac{M}{\alpha(1-\alpha)}\|f\|_{L^q([0,T];E_{\alpha,q})},$$

and, moreover, the constant M is independent of f, α, and q.

Theorem 13.4.6 ([82]). *Let $A \in \mathcal{H}(\omega,\theta)$, and let $1 < p, q < \infty$, or $p = q = \infty$. Then problem (13.1) admits an absolutely continuous solution $u(\cdot)$ such that $u', Au \in L^p([0,T];E_{\alpha,q})$, and $u(\cdot)$ is a continuous*

$E_{1+\alpha-\frac{1}{p},q}$*-valued function iff* $f(\cdot) \in L^p([0,T]; E_{\alpha,p})$ *and* $u^0 \in E_{1+\alpha-\frac{1}{p},q}$. *This solution* $u(\cdot)$ *satisfies the inequality*

$$\|u'(\cdot)\|_{L^p([0,T];E_{\alpha,q})} + \|Au(\cdot)\|_{L^p([0,T];E_{\alpha,q})} + \max_{0\le t\le T} \|u(t)\|_{E_{1+\alpha-\frac{1}{p},q}}$$
$$\le M\Big(\|u^0\|_{E_{1+\alpha-\frac{1}{p},q}} + \frac{M(p,q)}{\alpha(1-\alpha)}\|f\|_{L^p([0,T];E_{\alpha,q})}\Big),$$

where $M(p,q) = \dfrac{M(q)p^2}{p-1}$ *if* $p \neq q$ *and* $M(p,p) = 1$.

Theorem 13.4.7 ([82]). *Let* $p = q = 1$ *or* $p = q = \infty$. *Then problem* (13.1) *has an absolutely continuous solution* $u(\cdot)$ *such that* $u'(\cdot), Au(\cdot) \in L^p([0,T]; E_{\alpha,p})$ *iff* $f(\cdot) \in L^p([0,T]; E_{\alpha,q}))$ *and* $u^0 \in E_{1+\alpha-\frac{1}{p},}$.

This solution satisfies the inequality

$$\|u(\cdot)\|_{L^p([0,T];E_{\alpha,p})} + \|Au(\cdot)\|_{L^p([0,T];E_{\alpha,p})} \le M\left(\|u^0\|_{1+\alpha-\frac{1}{p},p} + \frac{1}{\alpha(1-\alpha)}\|f\|_{L^p([0,T];E_{\alpha,p})}\right).$$

Proposition 13.4.3 ([82]). *Let* $1 < p < \infty$, $f(\cdot) \in L^p([0,T]; E_{\alpha,p})$, *and let* $u^0 \in E_{1+\alpha-\frac{1}{p},\infty}$. *Then* $u(\cdot)$ *from formula* (13.2) *satisfies the estimate*

$$\max_{0\le t\le T} \|u(t)\|_{E_{1+\alpha-\frac{1}{p},\infty}} \le M\Big(\|u^0\|_{E_{1+\alpha-\frac{1}{p},\infty}} + \frac{p^2}{p-1}\|f\|_{L^p([0,T];E_{\alpha,\infty})}\Big).$$

We set $|\tilde{\Omega}| = \mu(\tilde{\Omega})$ for a μ-measurable set $\tilde{\Omega} \in \Omega$.

Definition 13.4.6. If $(\exp(tA)f)(x) = \int_\Omega k(t,x,y)f(y)\,dy$, $t \in \mathbb{R}$, then we say that k satisfies the *Poisson estimate of order* $m \in \mathbb{N}$ if $|k(t,x,y)| \le P(t,x,y)$ for almost all $x, y \in \Omega$, where

$$P(t,x,y) := |B(x,t^{1/m})|^{-1} p\left(\frac{d(x,y)^m}{t}\right)$$

and $p(\cdot)$ is a bounded, continuous, and strongly positive function satisfying the condition

$$\lim_{r\to\infty} r^{n+\delta} p(r^m) = 0$$

for a certain $\delta > 0$ and $|B(x,\rho)| := \{y \in \Omega : d(x,y) < \rho\}$.

Theorem 13.4.8 ([163]). *Let* $1 < p, q < \infty$ *and let* (Ω, μ, d) *be a topological space satisfying the following conditions*:

(i) $|B(x,2\rho)| \le C|B(x,\rho)|$, *where* $B(x,\rho)$ *is the ball of radius* ρ *centered at a point* x;

(ii) $\operatorname*{ess\,sup}_{x\in\Omega} |B(x,\rho)| \le C \operatorname*{ess\,inf}_{x\in\Omega} |B(x,\rho)|$;

(iii) *the operator* A *generates an analytic* C_0*-semigroup on* $L^2(\Omega)$ *with* $\omega(A) < 0$.

Let a semigroup $\exp(\cdot A)$ *be represented by a kernel satisfying the Poisson estimate of order* $m \in \mathbb{N}$*. Then* $A \in MR(p, L^q(\Omega))$*, i.e., for each* $f \in L^p(\overline{\mathbb{R}}_+; L^q(\Omega))$*, there exists a unique solution* $u \in W^{1,p}(\overline{\mathbb{R}}_+; L^q(\Omega)) \cap L^p(\overline{\mathbb{R}}_+; \mathcal{D}(A_q))$ *of problem* (13.1) *with* $u^0 = 0$ *in the sense of* $L^p(\overline{\mathbb{R}}_+; L^q(\Omega))$*. Moreover,*

$$\int_0^\infty \|u(t)\|^p_{L^q(\Omega)}\, dt + \int_0^\infty \|u'(t)\|^p_{L^q(\Omega)}\, dt + \int_0^\infty \|Au(t)\|^p_{L^q(\Omega)}\, dt \le C \int_0^\infty \|f(t)\|^p_{L^q(\Omega)}\, dt$$

for any $f(\cdot) \in L^p(\overline{\mathbb{R}}_+; L^q(\Omega))$*.*

Definition 13.4.7. A positive operator $A \in \mathcal{C}(E)$ is called an *operator of bounded imaginary powers* if there exist $\varepsilon > 0$ and $M \ge 1$ such that $A^{it} \in B(E)$ and $\|A^{it}\| \le M$ for $-\varepsilon \le t \le \varepsilon$.

Proposition 13.4.4 ([77])**.** *Let A be a positive operator of bounded imaginary powers. Then there exist* $M \ge 1$ *and* $\theta \ge 0$ *such that* $\{A^{it}\}_{t\in\mathbb{R}}$ *is a* C_0*-group of operators on* E *with the generator* $i \log A$ *and* $\|A^{it}\| \le Me^{\theta|t|}$, $t \in \mathbb{R}$*.*

Note that if $E = H$ is Hilbert and $A = A^* \ge \alpha I > 0$, then A is an operator of bounded imaginary powers.

Definition 13.4.8. Let $\mathcal{S}(\mathbb{R}; E)$ be the Schwartz space of smooth rapidly decreasing E-valued functions. For $u(\cdot) \in \mathcal{S}(\mathbb{R}; E)$, define the *Hilbert transform*

$$(Hu)(t) := \frac{1}{\pi}\,\mathrm{PV}\left(\frac{1}{t}\right) * u = \frac{1}{\pi}\mathrm{p.v.} \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau}\, d\tau, \quad t \in \mathbb{R}.$$

For an arbitrary Banach space E, the Hilbert transform can be not a bounded operator on $L^p(\mathbb{R}; E)$, even for a certain $p \in (1, \infty)$.

Definition 13.4.9. A space E is called an *UMD-space* if the Hilbert transform is a bounded operator on $L^p(\mathbb{R}; E)$ for a certain $p \in (1, \infty)$.

Proposition 13.4.5 ([77])**.** *A Hilbert space* H*, any Banach space isomorphic to an UMD-space, any interpolation space* $(X, Y)_{\theta,p}$ *and* $[X, Y]_{\theta,p}$ *constructed via an interpolation pair of UMD-spaces, and any finite-dimensional space are UMD-spaces.*

Proposition 13.4.6 ([77])**.** *Let* E *be an UMD-space. Then the Hilbert transform is a bounded operator on* $L^p(\mathbb{R}, E)$ *for any* $p \in (1, \infty)$*.*

Proposition 13.4.7 ([77])**.** *Let* E *be an UMD-space, and let an operator* $A \in \mathcal{H}(\omega, \theta)$ *be such that* $\|(-A)^{it}\| \le Me^{\phi|t|}$, $t \in \mathbb{R}$*. Then problem* (13.1) *is coercively solvable in* $L^p([0, T]; E)$ *with* $F = (E, D(A))_{1-\frac{1}{p},p}$*.*

In connection with Proposition 13.4.7, the following assertion is of interest.

Proposition 13.4.8 ([193]). *Let $A \in \mathcal{H}(0, \beta)$ on a Hilbert space $E = H$. The following conditions are equivalent*:

(i) *there exist $C > 0$ and ω such that $\|(-A)^{it}\| \le Ce^{\omega t},\ t \in \mathbb{R}$;*

(ii) *there exists an operator $Q \in B(H)$ such that $Q^{-1} \in B(H)$ and $\|Q^{-1}\exp(tA)Q\| \le 1,\ t \in \overline{\mathbb{R}_+}$.*

It is clear that the study of the coercivity of problems (13.1) is in fact that of the convolution operator $A\int\limits_0^t \exp((t-s)A)f(s)\,ds$ on the space $L^p([0,T];E)$. For such an operator, it is natural to apply the Mikhlin theorem on Fourier multipliers in order to prove its continuity on the $L^p(\mathbb{R};E)$ space. Recently, this approach was realized in [172, 292, 293].

The Poisson semigroup on $L^1(\mathbb{R})$ and on $L^p(\mathbb{R};E)$ is not coercively well posed on the $L^p(\mathbb{R},E)$ space if E is not an UMD-space (see [189]). Hence the assumption on E to be an UMD space is necessary in some sense.

But it was an open problem whether every generator of an analytic semigroup on $L^q(\Omega,\mu)$, $1 < q < \infty$, yields the coercive well-posedness in $L^p(\mathbb{R};E)$. Recently, Kalton and Lancien [171] gave a strong negative answer to this question. If every bounded analytic semigroup on a Banach space E is such that problem (13.1) is coercively well posed, then E is isomorphic to a Hilbert space.

If A generates a bounded analytic semigroup $\{\exp(zA) : |\arg(z)| \le \delta\}$ on a Banach space E, then the following three sets are bounded in the operator norm

(i) $\{\lambda(\lambda I - A)^{-1} : \lambda \in i\mathbb{R}, \lambda \ne 0\}$;

(ii) $\{\exp(tA), tA\exp(tA) : t > 0\}$;

(iii) $\{\exp(zA) : |\arg z| \le \delta\}$.

In Hilbert spaces, this already implies the coercive well-posedness in $L^p(\mathbb{R}_+;E)$, but only in Hilbert spaces E. The additional assumption that we need in more general Banach spaces E is the R-boundedness.

A set $\mathcal{T} \subset B(E)$ is said to be *R-bounded* if there exists a constant $C < \infty$ such that for all $Q_1, \ldots, Q_k \in \mathcal{T}$ and $x_1, \ldots, x_k \in E, k \in \mathbb{N}$,

$$\int_0^1 \left\| \sum_{j=0}^{k} r_j(u) Q_j(x_j) \right\| du \le C \int_0^1 \left\| \sum_{j=0}^{k} r_j(u) x_j \right\| du, \tag{13.13}$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables, e.g., the Rademacher functions $r_j(t) = \operatorname{sign}(\sin(2^j\pi t))$ on $[0,1]$. The smallest C such that (13.13) is fulfilled is called the R-boundedness constant of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$.

Theorem 13.4.9 ([293]). *Let A generate a bounded analytic semigroup $\exp(tA)$ on an UMD-space E. Then problem (13.1) is coercively well posed in the space $L^p(\mathbb{R}_+; E)$ iff one of sets (i), (ii), and (iii) presented above is R-bounded.*

A discrete variant of Theorem 13.4.9 was considered in [83].

Definition 13.4.10. Problem (13.5) is said to be *coercively solvable* in $L^p([0,T];E)$, $1 \le p \le \infty$, if for any $f(\cdot) \in L^p([0,T];E)$, there exists a unique solution $u(\cdot)$ satisfying the equation almost everywhere such that $u(0) = u^0$, $u'(0) = u^1$, $u''(\cdot), Au(\cdot) \in L^p([0,T];E)$, and the following coercive inequality holds:

$$\|u''(\cdot)\|_{L^p([0,T];E)} + \|Au(\cdot)\|_{L^p([0,T];E)} \le M(p)(\|f(\cdot)\|_{L^p([0,T];E)} + \|u^0\|_{D(A)} + \|u^1\|_{E^1}). \tag{13.14}$$

Theorem 13.4.10 ([232]). *Let problem (13.5) be coercively solvable in $L^p([0,T];E)$ with a certain $1 \le p \le \infty$. Then A is bounded.*

Proof. For simplicity, we set $u^0 = u^1 = 0$. Then (13.14) implies that the operator $(Kf)(t) := A\int_0^t S(t-s,A)f(s)\,ds$, i.e., the operator K from (12.19) with $B = I$, is a continuous operator acting from $L^p([0,T];E)$ into $L^p([0,T];E)$. Theorem 12.7.1 implies $K \in B(L^p([0,T];E), C([0,T];E))$. We now take $f \in C([0,T];E)$. Then we obtain from Corollary 12.8.2 that $\|AS(\cdot,A) * f\|_{C([0,T];E)} \le CT^{1/p}T^{1/q}\|f\|_{C([0,T];E)}$, where $\frac{1}{p} + \frac{1}{q} = 1$, and, therefore, $\mathrm{SV}(C(\cdot,A),t) \le Ct$ as $t \to 0$. Therefore, by Proposition 8.1.14, we obtain the boundedness of the operator A.

13.5. Coercivity in $B([0,T]; C^{2\theta}(\overline{\Omega}))$

In [153], the following result was proved.

Theorem 13.5.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$ lying to one side of its topological boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of dimension $n-1$ and class $C^{2+\theta}$ for some $\theta \in (0,2) \setminus \{1\}$. Let $\mathcal{A} = \mathcal{A}(x, D_x) = \sum_{|\alpha|\le 2} a_\alpha(x) D_x^\alpha$ be a second-order strongly elliptic operator (i.e., $\mathrm{Re} \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha \ge \nu|\xi|^2$ for some $\nu > 0$ and for any $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$) with coefficients of class $C^\theta(\overline{\Omega})$. Then there exist $\mu \ge 0$, $\phi_0 \in \left(\frac{\pi}{2}, \pi\right)$ such that for any $\lambda \in \mathbb{C}$ with $|\lambda| \ge \mu$ and $|\operatorname{Arg}\lambda| \le \phi_0$, the problem*

$$\begin{cases} \lambda u - \mathcal{A}u = f, \\ \gamma_0 u = 0, \end{cases} \tag{13.15}$$

has a unique solution $u(\cdot)$ belonging to $C^{2+\theta}(\overline{\Omega})$ for any $f(\cdot) \in C^\theta(\overline{\Omega})$, and for a certain $M > 0$,

$$|\lambda|^{1+\frac{\theta}{2}}\|u\|_{C(\overline{\Omega})} + |\lambda|\,\|u\|_{C^\theta(\overline{\Omega})} + \|u\|_{C^{2+\theta}(\overline{\Omega})} \le M\Big(\|f\|_{C^\theta(\overline{\Omega})} + |\lambda|^{\frac{\theta}{2}}\|\gamma_0 f\|_{C(\partial\Omega)}\Big), \tag{13.16}$$

where γ_0 is the trace operator on $\partial\Omega$.

It is clear from (13.16) that operator $\mathcal{A}$ does not generate C_0-semigroup on $E = C^\theta(\overline{\Omega})$ space in general, but, following, say, to [201], we can construct a semigroup $\exp(t\mathcal{A}), t \geq 0$, which is analytic.

Let us consider the following mixed Cauchy–Dirichlet parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \mathcal{A}u(t,x) + f(t,x), & t \in [0,T], x \in \overline{\Omega}, \\ u(t,x') = g(t,x'), & t \in [0,T], x' \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{13.17}$$

Definition 13.5.1. We say that problem (13.17) has a strict solution if there exists a continuous function $u(t,x)$ such that it has the first derivative in t and the derivatives of order less than or equal to 2 in the space variables that are continuous up to the boundary of $[0,T] \times \overline{\Omega}$, i.e., $u(\cdot) \in C^1\Big([0,T]; C(\overline{\Omega})\Big) \cap C\Big([0,T]; C^2(\overline{\Omega})\Big)$, and the equations in (13.17) are satisfied.

The space $B([0,T]; C^{2\theta}(\overline{\Omega}))$ is defined as the space of bounded functions $u(\cdot) : [0,T] \to C^{2\theta}(\overline{\Omega})$ endowed with the usual sup-norm.

Theorem 13.5.2 ([153]). *Let the following assumptions hold for some $\theta \in (0,2) \setminus \{1\}$:*

(I) *Ω is an open bounded subset of $\mathbb{R}^n$ lying to one side of its topological boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of dimension $n-1$ and class $C^{2+2\theta}$;*

(II) *operator $\mathcal{A} = \mathcal{A}(x, \partial_x) = \sum\limits_{|\alpha|\leq 2} a_\alpha(x)\partial_x^\alpha$ is a second-order strongly elliptic operator (i.e., $\operatorname{Re} \sum\limits_{|\alpha|=2} a_\alpha(x)\xi^\alpha \geq \nu|\xi|^2$ for some $\nu > 0$ and for any $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^n$) with coefficients of class $C^{2\theta}(\overline{\Omega})$.*

Then problem (13.17) has a unique strict solution $u(\cdot)$ belonging to $B\Big([0,T]; C^{2+2\theta}(\overline{\Omega})\Big)$ such that $\dfrac{\partial u}{\partial t} \in B\Big([0,T]; C^{2\theta}(\overline{\Omega})\Big)$ iff the following conditions are satisfied:

(a) $u_0 \in C^{2+2\theta}(\overline{\Omega})$;

(b) $f \in C\Big([0,T]; C(\overline{\Omega})\Big) \cap B\Big([0,T]; C^{2\theta}(\overline{\Omega})\Big)$;

(c) $g \in B\Big([0,T]; C^{2+2\theta}(\partial\Omega)\Big) \cap C\Big([0,T]; C^2(\partial\Omega)\Big) \cap C^1\Big([0,T]; C(\partial\Omega)\Big)$, $\dfrac{\partial g}{\partial t} \in B\Big([0,T]; C^{2\theta}(\partial\Omega)\Big)$, $\dfrac{\partial g}{\partial t} - \gamma f \in C^\theta\Big([0,T]; C(\partial\Omega)\Big)$;

(d) $\gamma u_0 = g(0,\cdot)$;

(e) $\dfrac{\partial g}{\partial t}(0,\cdot) - \gamma f(0,\cdot) = \gamma \mathcal{A} u_0$.

13.6. Boundary Value Problems

Let us consider the following two-point problem:

$$\begin{cases} u^{(m)}(t) = Au(t) + f(t), \quad t \in [0,T], \\ u^{(j)}(0) = u_j^0, \ j \in \alpha_1, \quad u^{(k)}(T) = u_k^1, \quad k \in \alpha_2, \end{cases} \tag{13.18}$$

with a continuous function $f(\cdot) \in C([0,T];E)$.

By a solution of problem (13.18), we mean a function $u(\cdot) \in C^m([0,T];E)$ taking its values in $D(A)$ and satisfying (13.18) with $u_j^0,\ u_k^1 \in D$ for a certain set D dense in $D(A)$.

Obviously, the definition of well-posedness is as follows: if $f_n(t) \to 0$ uniformly in t from the closed interval $[0,T]$ and $u_{j,n}^0 \to 0,\ u_{k,n}^1 \to 0$ as $j \in \alpha_1,\ k \in \alpha_2$, and $n \to \infty$, then $u_n(t) \to 0$ uniformly in $t \in [0,T]$.

Theorem 13.6.1 ([136]). *Let problem* (13.18) *be well posed for* $u_j^0 = 0,\ u_k^1 = 0$ *for any* $j \in \alpha_1, k \in \alpha_2$. *Then* $m_0 + m_1 \geq m$, *where* $m_0 = |\alpha_0|,\ m_1 = |\alpha_1|$.

Theorem 13.6.2 ([136]). *Let problem* (13.18) *be well posed for* $f(\cdot) \equiv 0$. *Then* $m_0 + m_1 \leq m$.

Theorem 13.6.3 ([136]). *Let problem* (13.18) *be well posed, and let either*

(i) m *be even and* $m_0 < \dfrac{m-2}{2}$ *or* $m_1 < \dfrac{m-2}{2}$,

or

(ii) m *be odd and* $m_0 < \dfrac{m-1}{2}$ *or* $m_1 < \dfrac{m-1}{2}$.

Then A *is bounded.*

Theorem 13.6.4 ([106]). *Let* $A \in \mathcal{C}(M,\omega)$ *and* $-\mathbb{N}_0^2 \subseteq \rho(A)$, *and let both limits* (5.1) *exist for all* $x \in E$. *Then there exists a unique solution of the Dirichlet problem*

$$u''(t) = Au(t), \quad u(0) = x, \quad u(\pi) = y, \quad 0 \leq t \leq \pi,$$

and, moreover,

$$\sup_{0 \leq s \leq \pi} \|u(s)\| \leq c(\|u(0)\| + \|u(\pi)\|).$$

Let us consider in the Banach space E the boundary value problem

$$u''(t) = Au(t) + f(t),\, t \in (0,T), u(0) = u^0, u(T) = u^T, \tag{13.19}$$

where operator $-A$ generates a C_0-semigroup and $f(\cdot)$ is some function from $[0,T]$ to E. Problem (13.19) can be considered in different functional spaces.

A function $u(t)$ is called a solution of the elliptic problem (13.19) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the interval [0,T]. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives;

(ii) the element $u(t)$ belongs to $D(A)$ for all $t \in [0,T]$ and the function $Au(t)$ is continuous on the segment [0,T];

(iii) $u(t)$ satisfies the equation and boundary conditions (13.19).

A solution of problem (13.19) defined in this manner will from now on be referred to as a solution of problem (13.19) in the space $C([0,T],E)$.

The coercive well-posedness in $C([0,T],E)$ of the boundary value problem (13.19) means that the coercive inequality

$$\|u''\|_{C([0,T],E)} + \|Au\|_{C([0,T],E)} \leq M\Big(\|f\|_{C([0,T],E)} + \|Au^0\|_E + \|Au^T\|_E\Big)$$

holds for its solution $u(\cdot) \in C([0,T],E)$ with some M which is independent of u^0, u^T, and $f(t) \in C([0,T],E)$.

It turns out that the positivity of the operator A in E is a necessary condition of the coercive well-posedness of boundary value problem (13.19) in $C([0,T],E)$. Is the positivity of the operator A in E a sufficient condition for the coercive well-posedness of boundary value problem (13.19)? In general case, the answer is negative.

The coercive well-posedness of boundary value problem (13.19) was established in

$$C_{0T}^{\beta,\gamma}([0,T];E), \quad (0 \leq \gamma \leq \beta, 0 < \beta < 1),$$

the space obtained by completion of the space of all smooth E-valued functions $\varphi(t)$ on [0,T] in the norm

$$\|\varphi\|_{C_{0T}^{\beta,\gamma}([0,T];E)} = \max_{0\leq t\leq T} \|\varphi(t)\|_E + \sup_{0\leq t<t+\tau\leq T} \frac{(t+\tau)^\gamma (T-t)^\gamma \|\varphi(t+\tau)-\varphi(t)\|_E}{\tau^\beta}.$$

Theorem 13.6.5 ([1]). *Let A be the positive operator in a Banach space E and $f(\cdot) \in C_{0T}^{\beta,\gamma}([0,T];E)$ $(0 \leq \gamma \leq \beta, 0 < \beta < 1)$. Then for the solution $u(t)$ in $C_{0T}^{\beta,\gamma}([0,T];E)$ of the boundary value problem* (13.19), *the coercive inequality*

$$\|u''\|_{C_{0T}^{\beta,\gamma}([0,T];E)} + \|Au\|_{C_{0T}^{\beta,\gamma}([0,T];E))} + \|u''\|_{C([0,T];E_{\beta-\gamma})}$$
$$\leq M\left(\|f(0)-Au^0\|_{E_{\beta-\gamma}} + \|f(T)-Au^T\|_{E_{\beta-\gamma}} + \beta^{-1}(1-\beta)^{-1}\|f\|_{C_{0T}^{\beta,\gamma}([0,T];E)}\right)$$

holds, where M is independent of β, γ, u^0, u^T, and $f(t)$.

Recall that here the Banach space $E_\alpha, 0 < \alpha < 1$, consists of those $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha}\|A^{\frac{1}{2}}\exp(-zA^{\frac{1}{2}})v\|_E + \|v\|_E$$

is finite.

Theorem 13.6.6 ([1]). *Let A be the positive operator in a Banach space E and*

$$f(t) \in C_{0T}^{\beta,\gamma}([0,T];E_{\alpha-\gamma}), \quad (0 \leq \gamma \leq \beta \leq \alpha, \;\; 0 < \alpha < 1).$$

Then for the solution $u(\cdot) \in C^{\beta,\gamma}_{0T}([0,T];E_{\alpha-\gamma})$ *of boundary value problem* (13.19), *the coercive inequality*

$$\|u''\|_{C^{\beta,\gamma}_{0T}([0,T];E_{\alpha-\beta})} + \|Au\|_{C^{\beta,\gamma}_{0T}([0,T];E_{\alpha-\beta})} + \|u''\|_{C([0,T];E_{\alpha-\gamma})}$$
$$\leq M\left(\|f(0) - Au^0\|_{E_{\alpha-\gamma}} + \|f(T) - Au^T\|_{E_{\alpha-\gamma}} + \alpha^{-1}(1-\alpha)^{-1}\|f\|_{C^{\beta,\gamma}_{0T}([0,T];E_{\alpha-\beta})}\right)$$

holds, where M *is independent of* $\alpha, \beta, \gamma, u^0, u^T$, *and* $f(t)$.

Theorem 13.6.7 ([71]). *Let* A *be a strongly positive operator in a Banach space* E *and let problem* (13.19) *be coercive well-posed in* $L^{p_0}([0,T];E)$ *for some* $1 < p_0 < \infty$. *Then it is also coercive well-posed in* $L^p([0,T];E)$ *for any* $1 < p < \infty$ *and*

$$\|Au(\cdot)\|_{L^p([0,T];E)} + \max_{0\leq t\leq T}\|u(t)\|_{E_{1-1/p}} \leq \frac{Mp^2}{p-1}\big(\|\varphi_n\|_{L^p([0,T];E)} + \|u(0)\|_{E_{1-1/p}} + \|u(T)\|_{E_{1-1/p}}\big).$$

Here the space E_α coincides with an equivalent norm with the real interpolation space $(E, D(A^{\frac{1}{2}}))_{1-1/p,p}$.

Consider the problem

$$u''(t) = Au(t) + f(t),\ t \in \mathbb{R} \tag{13.20}$$

with bounded solutions and a sectorial operator A on E, i.e., $-A \in H(0,\theta)$.

Theorem 13.6.8 ([292]). *Let* A *be given on an UMD-space* E. *Then problem* (13.20) *is coercively solvable*:

$$\|u''\|_{L^p(\mathbb{R};E)} + \|Au(\cdot)\|_{L^p(\mathbb{R};E)} \leq c\|f\|_{L^p(\mathbb{R};E)} \ \textit{for any}\ f \in L^p(\mathbb{R};E),$$

iff the set $\{\lambda(\lambda I - A)^{-1} : \lambda < 0\}$ *is R-bounded in* $B(E)$.

In the space E, let us consider the problem

$$u''(t) = Au(t) + f(t), \quad t \in [0,T],$$
$$L_i u = \alpha_{i_1}u(0) + \alpha_{i_2}u'(0) + \beta_{i_1}u(T) + \beta_{i_2}u'(T) = f_i,\ i = 1,2, \tag{13.21}$$

with a positive operator A. The boundary value problem (13.21) is said to be uniformly well-posed in $X \subset E$ and $[a,b] \subset [0,T]$ if for any $f_1, f_2 \in E$, its solution $u(\cdot) \in C^2([0,T];E)$ exists, is unique, and is stable with respect to $f_i, i = 1,2$, uniformly in $t \in [0,T]$, i.e.,

$$\sup_{t\in[a,b]}\|u(t) - \tilde{u}(t)\| \leq C\big(\|f_1 - \tilde{f}_1\| + \|f_2 - \tilde{f}_2\|\big).$$

In [30, 47], necessary and sufficient conditions under which problem (13.21) with a positive operator A is uniformly well-posed are given. Also, in these papers, the authors present a number of theorems on the well-posedness and ill-posedness of elliptic problems and also theorems for problems of the form

$$u'(t) = Au(t), \quad 0 < t < T, \quad \mu u(0) + u(T) = u^0.$$

Chapter 14

SEMILINEAR PROBLEMS

At present, there is a sufficiently large material (see, e.g., [48, 95, 97]) devoted to studying nonlinear problems $u'(t) = (Au)(t)$ and $u'(t) \in (Au)(t)$ by using semigroup methods. In this chapter, we consider only problems with a linear principal operator A and a smooth nonlinear right-hand side f. Namely for these semilinear problems, the numerical analysis is sufficiently well elaborated, which we present in two theorems only. For general approximation theorems and the numerical analysis, see the first article in this volume.

14.1. First Order Equation

In a Banach space E, let us consider the following Cauchy problem:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in [0, T], \\ u(0) = u^0, \end{cases} \tag{14.1}$$

with an operator A generating a C_0-semigroup. Here, the function $f : [0, T] \times E \to E$. The existence and uniqueness of solutions of problem (14.1) is discussed in detail, e.g., in [74]. A classical solution of problem (14.1) is defined analogously to Definition 13.2.1.

Note that each classical solution of problem (14.1) satisfies the equation

$$u(t) = (Ku)(t) \equiv \exp(tA)u^0 + \int_0^t \exp((t-s)A) f(s, u(s))\, ds. \tag{14.2}$$

Definition 14.1.1. A continuous solution $u(\cdot)$ of Eq. (14.2) is called a *mild solution* of problem (14.1).

It is clear that in the semilinear case, as in the case of inhomogeneous equations, a mild solution can be not a classical solution.

Theorem 14.1.1 ([20]). *Let $A \in \mathcal{G}(M, \omega)$, $u^0 \in E$, a function $f : \overline{\mathbb{R}}_+ \times E \to E$ be continuous in t, and let it be Lipschitz in the second argument, i.e., for each $\tau > 0$,*

$$\|f(t, x) - f(t, y)\| \le L(\tau)\|x - y\|_E$$

for any $x, y \in E$, $0 \leq t \leq \tau$, *and a certain constant* $L(\tau)$. *Then problem* (14.1) *has a unique mild solution on* $\overline{\mathbb{R}}_+$.

If the Lipschitz condition holds locally, then a local existence theorem of mild solutions holds.

Conditions for the existence and uniqueness for problems of form (14.1)–(14.2) were studied in detail, e.g., in [20, 74, 75, 77, 158].

Proposition 14.1.1 ([46]). *Let* $-A$ *be strongly positive, and let* $A^{-1} \in B_0(E)$. *Let* $f(t, u)$ *be continuous in totality of the variables, and let* $\|f(t, u(t))\| \leq c(R) < \infty$ *for* $t \in [0, t_0]$ *and* $\|u\| \leq R$. *Then there exists at least one mild solution of problem* (14.1) *on* $[0, t^*] \subset [0, t_0]$, $t^* \leq t_0$.

Proposition 14.1.2 ([46]). *Let* $-A$ *be strongly positive. Let* $f(t, A^{-\alpha}w)$ *be continuous on* $[0, t_0]$ *for any* $w \in E$, *and let* $\|f(t, A^{-\alpha}w_1) - f(t, A^{-\alpha}w_2)\| \leq c(R)\|w_1 - w_2\|$, $\|w_1\|, \|w_2\| \leq R$. *Finally, let* $u^0 \in D(A^\alpha)$. *Then there exists a unique mild, i.e., continuous solution* $w(\cdot)$ *of the equation*

$$w(t) = \exp(tA)A^\alpha u^0 + \int_0^t A^\alpha \exp((t-s)A) f(s, A^{-\alpha}w(s))ds$$

defined on $[0, t^*] \subset [0, t_0]$.

N. Pavel [233] gave a necessary and sufficient condition for the existence of a local solution of (14.1) in case of an analytic semigroup.

Theorem 14.1.2 ([233]). *Let* $D \subseteq E$ *be a locally closed subset,* $f(\cdot) : [0, \tau) \to E$ *be a continuous functional, and let* $\exp(\cdot A) \in B_0(E)$. *A local classical solution* $u(\cdot) : [0, \tau) \to D$ *of* (14.1) *exists if and only if for all* $z, u^0 \in D$,

$$\lim_{h \to 0} \operatorname{dist}(\exp(hA)z + hf(t, z); D) = 0.$$

Let Ω be an open set in a Banach space E, and let $\mathcal{B} : \overline{\Omega} \to E$ be a compact operator having no fixed points on the boundary of Ω. Then for the vector field $\mathcal{W}(x) = x - \mathcal{B}x$, the rotation (degree) $\gamma(I - \mathcal{B}; \partial\Omega)$ being integer-valued characteristics of this field is well defined. Let z^* be an isolated fixed point of the operator $\mathcal{B}$ in the ball $B(z^*, r_0)$ of radius r_0 centered at a point z^*. Then $\gamma(I - \mathcal{B}; \partial B(z^*, r_0)) = \gamma(I - \mathcal{B}; \partial B(z^*, r))$ for $0 < r < r_0$.

This common value of the rotation is called the index of the fixed point z^* and is denoted by $\operatorname{ind}(z^*; I - \mathcal{B})$.

Theorem 14.1.3 ([26]). *Let* $A \in \mathcal{H}(\omega, \theta)$, *the resolvent* $(\lambda I - A)^{-1}$ *be compact for a certain* $\lambda \in \rho(A)$, *and let the operator* K *be given by formula* (14.2). *If* $u^*(\cdot)$ *is a unique mild solution of problem* (14.1), *then* $\operatorname{ind}(u^*(\cdot); I - K) = 1$.

This theorem is used, e.g., in approximating the Cauchy problem (14.1) with respect to the space, as well as with respect to time [60].

Definition 14.1.2. A solution of the Cauchy problem (14.1) is said to be *Lyapunov stable* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that the inequality $\|u(0) - \tilde{u}(0)\| \leq \delta$ implies $\max\limits_{0 \leq t < \infty} \|u(t) - \tilde{u}(t)\| \leq \varepsilon$ where $\tilde{u}(\cdot)$ is a solution of problem (14.1) with the initial condition $\tilde{u}(0)$.

Definition 14.1.3. A solution of the Cauchy problem (14.1) is said to be *uniformly asymptotically stable* at a point $u(0)$ if it is Lyapunov stable, and for any mild solution $\tilde{u}(\cdot)$ of problem (14.1) with $\|u(0)-\tilde{u}(0)\| \leq \delta$, it follows that $\lim\limits_{t\to\infty} \|u(t)-\tilde{u}(t)\| = 0$ uniformly in $\tilde{u}(0) \in B(u(0), \delta)$, i.e., there exists a function $\Phi_{u(0),\delta}(\cdot)$ such that $\|u(t) - \tilde{u}(t)\| \leq \Phi_{u(0),\delta}(t)$ with $\Phi_{u(0),\delta}(t) \to 0$ as $t \to \infty$.

We note that conditions for the existence of uniform asymptotic stability of a solution of problem (14.1) are given, e.g., in [74, Theorem 8.1.8]. These conditions are related to the location of the spectrum of the operator $A + \dfrac{\partial}{\partial u} f(t, u^*(\cdot))$.

In a Banach space E, let us consider the following periodic problem:

$$v'(t) = Av(t) + f(t, v(t)), \quad v(0) = v(T), \quad t \in \overline{\mathbb{R}}_+, \tag{14.3}$$

with an operator $A \in \mathcal{H}(\omega, \theta)$. In the case of periodic solutions, an analog of Eq. (14.2) is the integral equation

$$\begin{aligned} v(t) = (Kv)(t) &\equiv \exp(tA)(I + \exp(TA))^{-1} \int_0^T \exp((T-s)A) f(s, v(s))\, ds \\ &+ \int_0^t \exp((t-s)A) f(s, v(s))\, ds, \quad t \in [0, T]. \end{aligned} \tag{14.4}$$

As was noted in [17, Proposition 2.1.36], for the solvability of problem (14.3), it suffices to assume the existence of an inverse operator $(I - \exp(tA))^{-1}$ for $t > t_0$. Then $(I - \exp(tA))^{-1} \in B(E)$ for any $t > 0$.

Denote by $u(\cdot, u^0)$ a solution of problem (14.1) with $u(0) = u^0$. Then the function $u(\cdot, u^0)$ satisfies Eq. (14.2), and we can define the shift operator $\mathcal{K}u^0 = u(T, u^0)$ along trajectories, which maps E into E. If $u(\cdot, u^0)$ is a T-periodic solution of problem (14.1), then u^0 is a zero of the vector field of the operator $\mathcal{K}$, i.e., $(I - \mathcal{K})(u^0) = 0$.

We call attention to that the operator K maps $C([0, T]; E)$ into $C([0, T]; E)$ and its fixed points, if they exist, are solutions of Eq. (14.4).

Theorem 14.1.4 ([96])**.** *Let $A \in \mathcal{H}(\omega, \theta)$, the resolvent $(\lambda I - A)^{-1}$ be compact for a certain $\lambda \in \rho(A)$, and let a function f be sufficiently smooth, so that there exists a periodic solution $v^*(\cdot)$ of problem* (14.3) *such that problem* (14.1) *has an isolated uniformly asymptotically stable solution at the point $u(0) = v^*(0)$. Then* $\mathrm{ind}(v^*(0); I - \mathcal{K}) = \mathrm{ind}(v^*(\cdot); I - K)$.

Proof. Let $S \equiv S(x^*, \rho)$. Then the rotation $\gamma(I - \mathcal{K}; \partial S)$ of the field $I - \mathcal{K}$ on the sphere ∂S is equal to the index $\operatorname{ind}(x^*; I - \mathcal{K})$:

$$\gamma(I - \mathcal{K}; \partial S) = \operatorname{ind}(x^*; I - \mathcal{K}). \tag{14.5}$$

Let

$$M = \sup_{x \in S} \max_{0 \le t \le T} \|v(t; x)\|, \tag{14.6}$$

and consider the domain $\Omega \subset F = C([0, T]; E)$ defined by

$$\Omega = \{u(\cdot) \in C([0, T]; E) : u(0) \in S,\ \|u(\cdot)\|_F \le M + 1\}.$$

The function $v^*(\cdot)$ is a unique zero of the compact vector field $I - K$ on $\overline{\Omega}$. Hence

$$\gamma(I - K; \partial\Omega) = \operatorname{ind}(v^*(\cdot); I - K).$$

In view of (14.5) and (14.6), to prove the contiguous theorem, it suffices to show that

$$\gamma(I - \mathcal{K}; \partial S) = \gamma(I - K; \partial\Omega). \tag{14.7}$$

For this purpose, on $\partial\Omega$, we consider the following family of compact vector fields:

$$\begin{aligned}\Phi(v(\cdot); \lambda) = v(t) - (1 - \lambda) \exp(tA) \Big(I - \exp(TA)\Big)^{-1} \int_0^T \exp\big((T - s)A\big) f\big(s; v(s)\big)\, ds \\ - \lambda \exp(tA) \mathcal{K}\big(v(0)\big) - \int_0^t \exp\big((t - s)A\big) f\big(s, v(s)\big)\, ds \quad (0 \le \lambda \le 1).\end{aligned} \tag{14.8}$$

The fields $\Phi\big(v(\cdot); \lambda\big)$ are nondegenerate on $\partial\Omega$. Indeed, if for certain $v_0(\cdot) \in \partial\Omega$ and $\lambda_0 \in [0, 1]$, we have $\Phi\big(v_0(\cdot); \lambda_0\big) = 0$, then

$$v_0(0) = (1 - \lambda_0) \Big(I - \exp(TA)\Big)^{-1} \int_0^T \exp\big((T - s)A\big) f\big(s, v_0(s)\big)\, ds + \lambda_0 v_0(T). \tag{14.9}$$

Since relation (14.9) and the property that $\Phi\big(v_0(\cdot); \lambda_0\big) = 0$ imply that the function $v_0(\cdot)$ is a mild solution of (14.1), we obtain

$$\int_0^T \exp\big((T - s)A\big) f\big(s, v_0(s)\big)\, ds = v_0(T) - \exp(TA) v_0(0). \tag{14.10}$$

Without loss of generality, we may set $\operatorname{Re} \sigma(A) < 0$. But (14.9) and (14.10) imply

$$\exp(TA)\big((v_0(0) - v_0(T)\big) = \lambda_0^{-1}\big(v_0(0) - v_0(T)\big).$$

If $v_0(0) - v_0(T) \neq 0$, then this element is an eigenvector of the operator $\exp(TA)$ with the eigenvalue $\lambda_0^{-1} > 1$. However, this is impossible, since $\operatorname{Re}\sigma(A) < 0$, and for analytic C_0-semigroups, $\sigma\big(\exp(TA)\big) \setminus \{0\} = e^{T\sigma(A)}$. Hence $v_0(0) = v_0(T)$, which implies that $v_0(\cdot)$ is a T-periodic solution of problem (14.3) and that it is a zero of the field $I - K$. We arrive at a contradiction.

The fields of families (14.8) are nondegenerate on $\partial\Omega$. Therefore, the fields $\Phi\big(v(\cdot);0\big) = I - K$ and $\Phi\big(v(\cdot);1\big)$ are homotopic on $\partial\Omega$. We obtain

$$\gamma(I - K; \partial\Omega) = \gamma\Big(\Phi\big(v(\cdot);1\big); \partial\Omega\Big). \tag{14.11}$$

On $\partial\Omega$, let us consider the following family of vector fields: $(0 \leq \lambda \leq T)$

$$\Psi\big(v(\cdot);\lambda\big) = v(t) - P_\lambda\Big(\exp(tA)\mathcal{K}\big(v(0)\big) + \int_0^t \exp\big((t-s)A\big) f\big(s, v(s)\big)\, ds\Big), \tag{14.12}$$

with the operator $P_\lambda : F \to F$ defined by $(P_\lambda w)(t) = w(t)$ for $0 \leq t \leq \lambda$ and $(P_\lambda w)(t) = w(\lambda)$ for $\lambda \leq t \leq T$.

The operator $Q(v(\cdot))(t) = \exp(tA)\mathcal{K}\big(v(0)\big) + \int_0^t \exp\big((t-s)A\big) f\big(s, v(s)\big)\, ds$, which maps F onto F, is compact. The operator $P_\lambda : F \to F$ is strongly continuous in λ. Therefore, the operator $P_\lambda Q$ is uniformly continuous in λ, and family (14.12) is a compact deformation (see [45, Sec. 19.1]).

Let us show that families (14.12) are nondegenerate on $\partial\Omega$. Assume that for certain $\lambda_0 \in [0,1]$ and $v_0(\cdot) \in \partial\Omega$, we have $v_0(\cdot) \neq v(\cdot; x^*)$ and $\Psi\big(v_0(\cdot);\lambda_0\big) = 0$. The boundary $\partial\Omega$ of the domain Ω consists of two parts

$$G_0 = \{v(\cdot) \in C([0,T];E) : v(0) \in S,\ \|v(\cdot)\|_{C([0,T];E)} = M + 1\}$$

and

$$G_1 = \{v(\cdot) \in C([0,T];E) : v(0) \in \partial S,\ \|v(\cdot)\|_{C([0,T];E)} \leq M + 1\}.$$

Let $v_0(\cdot) \in G_0$. Then

$$\|v_0(\cdot)\|_{C([0,T];E)} = M + 1. \tag{14.13}$$

On the other hand, since the function $v_0(\cdot)$ is a solution of (14.1) on the closed interval $[0,\lambda]$ and $v_0(0) \in S$, it follows from (14.6) that

$$\|v_0(\cdot)\|_{C([0,T];E)} = \max_{0 \leq t \leq T} \|v_0(t)\|_E = \max_{0 \leq t \leq \lambda} \|v_0(t)\|_E \leq M. \tag{14.14}$$

Equations (14.13) and (14.14) contradict one another. Therefore, there is only one possibility: $v_0(\cdot) \in G_1$ and $v_0(0) \in \partial S$. But $\Psi\big(v_0(\cdot);\lambda_0\big) = 0$ implies $v_0(0) = \mathcal{K}(v_0(0))$, which is impossible by the choice of the radius ρ of the ball S. Therefore, fields (14.12) are nondegenerate on $\partial\Omega$ and homotopic. Therefore, $\gamma\Big(\Psi\big(v(\cdot);0\big);\partial\Omega\Big) = \gamma\Big(\Psi\big(v(\cdot);T\big);\partial\Omega\Big)$. But $\Psi\big(v(\cdot);T\big) = \Phi\big(v(\cdot);1\big)$, and hence

$$\gamma\Big(\Psi\big(v(\cdot);0\big);\partial\Omega\Big) = \gamma\Big(\Phi\big(v(\cdot);1\big);\partial\Omega\Big). \tag{14.15}$$

Consider the vector field

$$\Psi\big(v(\cdot);0\big) = v(t) - \mathcal{K}\big(v(0)\big) \quad (v(\cdot) \in \partial\Omega).$$

Since the operator $\mathcal{K}\big(v(0)\big)$ can also be considered as a mapping from F into the space of constant functions, which is denoted by $\widetilde{E}$, its rotation (see [45]) coincides with the rotation of its restriction $\widetilde{\Psi}$ to $\partial\Omega \cap \widetilde{E}$. But the field $\widetilde{\Psi}\big(v(\cdot);0\big)$ on $\partial\Omega \cap \widetilde{E}$ is isomorphic to the field $I - \mathcal{K}$ on ∂S. Therefore,

$$\gamma\Big(\Psi\big(v(\cdot);0\big);\partial\Omega\Big) = \gamma\Big(\widetilde{\Psi}\big(v(\cdot);0\big);\partial\Omega \cap \widetilde{E}\Big) = \gamma(I - \mathcal{K};\partial S). \tag{14.16}$$

From (14.11), (14.15), and (14.16), we obtain (14.7). The theorem is proved.

To show how one can use, e.g., Theorem 14.1.4 in practice, let us define the following conditions.

(A) *Consistency.* There exists $\lambda \in \rho(A) \cap \bigcap_n \rho(A_n)$ such that the resolvents converge:

$$R(\lambda;A_n) \to R(\lambda;A).$$

(B) *Stability.* There are some constants $M_1 \geq 1$ and ω such that

$$\|R(\lambda;A_n)\| \leq M/|\lambda - \omega| \text{ for } \operatorname{Re}\lambda > \omega.$$

To formulate the convergence theorem, we need the following notation. By a semidiscrete approximation of the T-periodic problem (14.3), we mean the T-periodic problems

$$v_n'(t) = A_n v_n(t) + f_n\big(t, v_n(t)\big),\ v_n(t) = v_n(t+T), t \in \mathbb{R}_+, \tag{14.17}$$

where the operators A_n generate analytic semigroups in E_n, condition (A) is satisfied, the functions f_n are uniformly bounded: $\sup\limits_{t\in[0,T],\|x_n\|\leq c_1} \|f_n(t,x_n)\| \leq C_2$, the functions f_n approximate f and are sufficiently smooth with $f_n(t,x_n) = f_n(t+T,x_n)$ for any $x_n \in E_n$ and $t \in \mathbb{R}_+$.

The mild solutions of (14.17) are determined by the equations

$$\begin{aligned} v_n(t) = (K_n v_n)(t) \equiv \exp(tA_n)\Big(I_n - \exp(TA_n)\Big)^{-1} \int_0^T \exp\big((T-s)A_n\big) f_n\big(s, v_n(s)\big)\,ds \\ + \int_0^t \exp\big((t-s)A_n\big) f_n\big(s, v_n(s)\big)\,ds. \end{aligned} \tag{14.18}$$

Theorem 14.1.5 ([96]). *Assume that Conditions* (A) *and* (B) *hold and the compact resolvents* $R(\lambda; A)$ *and* $R(\lambda; A_n)$ *converge:*

$$R(\lambda; A_n) \to R(\lambda; A)$$

compactly for some $\lambda \in \rho(A)$*. Assume that*

(i) *the functions* f *and* f_n *are sufficiently smooth, so that there exists an isolated mild solution* $v^*(\cdot)$ *of periodic problem* (14.3) *with* $v^*(0) = x^*$ *such that the Cauchy problem*

$$u'(t) = Au(t) + f\big(t, u(t)\big),\ u(0) = x^*, \tag{14.19}$$

has a uniformly asymptotically stable isolated solution at the point x^**;*

(ii) $f_n(t, x_n) \to f(t, x)$ *uniformly in* $t \in [0, T]$ *as* $x_n \to x$*;*

(iii) *the space* E *is separable.*

Then for almost all n*, problems* (14.17) *have periodic mild solutions* $v_n^*(t), t \in [0, T]$*, in a neighborhood of* $p_n v^*(\cdot)$*, where* $v^*(\cdot)$ *is a mild periodic solution of* (14.3) *with* $v^*(0) = x^*$*. Each sequence* $\{v_n^*(\cdot)\}$ *is* $\mathcal{P}$*-compact, and* $v_n^*(t) \to v^*(t)$ *uniformly with respect to* $t \in [0, T]$*.*

Proof. We divide the proof into several steps.

Step 1. First, let us show that the compact convergence of resolvents $R(\lambda; A_n) \to R(\lambda; A)$ is equivalent to the compact convergence of C_0-semigroups $\exp(tA_n) \to \exp(tA)$ for any $t > 0$. Let $\|x_n\| = O(1)$. Then from the estimate $\|A_n \exp(tA_n)\| \le \dfrac{M}{t} e^{\omega t}$, we obtain the boundedness of the sequence $\{(A_n - \lambda) \exp(tA_n)x_n\}$. Because of the compact convergence of resolvents, we obtain the compactness of the sequence $\{\exp(tA_n)x_n\}$.

The necessity will be proved if for the measure of noncompactness $\mu(\cdot)$ (for the definition, see [278]), we establish that $\mu(\{(\lambda I - A_n)^{-1}x_n\}) = 0$ for $\|x_n\| = O(1)$. We have

$$\mu(\{(\lambda I - A_n)^{-1}x_n\}) = \mu\bigg(\bigg\{\int_0^\infty e^{-\lambda t} \exp(tA_n)x_n\bigg\}\bigg) \le \mu\bigg(\bigg\{\int_0^q e^{-\lambda t} \exp(tA_n)x_n dt\bigg\}\bigg)$$
$$+\mu\bigg(biggl\{\int_Q^\infty e^{-\lambda t} \exp(tA_n)x_n dt\bigg\}\bigg) + \mu\bigg(\bigg\{\exp(\epsilon A_n) \int_q^Q e^{-\lambda t} \exp\big((t-\epsilon)A_n\big)x_n dt\bigg\}\bigg).$$

Two first terms can be made less than ϵ by the choice of q, Q. The last term is equal to zero because of the compact convergence $\exp(\epsilon A_n) \to \exp(\epsilon A)$ for any $0 < \epsilon < q$.

Step 2. Consider the operators K and K_n defined by (14.4) and (14.18) on the spaces

$$F = C([0, T]; E) \equiv \{u(t) : \|u\|_F = \max_{t \in [0,T]} \|u(t)\|_E < \infty\}$$

and

$$F_n = C([0, T]; E_n) \equiv \{u_n(t) : \|u_n\|_{F_n} = \max_{t \in [0,T]} \|u_n(t)\|_{E_n} < \infty\}.$$

The operator K defined by (14.4) is compact in F. Indeed, we obtain that the operator

$$\mathcal{F}_\epsilon(u_k)(t) = \exp(\epsilon A)\int_0^{t-\epsilon} \exp\big((t-s-\epsilon)A\big) f\big(s, u_k(s)\big)ds$$

maps any bounded set of functions $\{u_k(\cdot)\}, \|u_k(\cdot)\|_F \leq C$, into a compact set in E for any $t > 0$ and $0 < \epsilon < t$. We see that $\|\mathcal{F}_\epsilon(u_k)(t) - \mathcal{F}(u_k)(t)\| \leq C\epsilon$ for any $t \in (0, T]$, where

$$\mathcal{F}(u_k)(t) = \int_0^t \exp\big((t-s)A\big) f\big(s, u_k(s)\big)ds$$

and $0 < \epsilon < t$. Then it follows that the operator $\mathcal{F}(\cdot)(t) : F \to E$ is compact for the same $t > 0$. For $t = 0$, the operator $\mathcal{F}(\cdot)(0)$ is also compact. Moreover, the set of functions $\{F_k(\cdot)\}, F_k(t) = \mathcal{F}(u_k)(t), t \in [0, T]$, is an equibounded and equicontinuous family, since for $0 < t_1 < t_2$, we obtain

$$\|F_k(t_2) - F_k(t_1)\| \leq C\Big(\int_0^{t_1} \|\exp\big((t_2-s)A\big) - \exp\big((t_1-s)A\big)\|ds + |t_2 - t_1|\Big),$$

and $\exp(\cdot A)$ is uniformly continuous in $t > 0$.

The sequence $\{y_k\}, y_k = \big(I - \exp(TA)\big)^{-1} \int\limits_0^T \exp\big((T-s)A\big) f\big(s, u_k(s)\big)ds \in E$, is compact, since $\{\mathcal{F}(u_k)(T)\}$ is a compact set. Therefore, $\{\exp(\cdot A)y_k\}$ is a compact sequence of functions in F. By the generalized Arzela–Ascoli theorem, it follows that operator K is compact.

Step 3. It is easy to see that $K_n \to K$. Indeed, $I_n \to I$ stably and $\exp(TA_n) \to \exp(TA)$ compactly; hence $I_n - \exp(TA_n) \to I - \exp(TA)$ regularly, the null space $\mathcal{N}\big(I - \exp(TA)\big) = \{0\}$ and the operators $I_n - \exp(TA_n)$ are Fredholm of index zero. Then it follows from [14] that $I_n - \exp(TA_n) \to I - \exp(TA)$ stably, i.e., $\big(I_n - \exp(TA_n)\big)^{-1} \to \big(I - \exp(TA)\big)^{-1}$ and the convergence $K_n \to K$ is a consequence of the dominated convergence theorem. To show that $K_n \to K$ compactly, we assume that $\|u_n\|_{F_n} = O(1)$. Now $\{K_n u_n\}$ is $\mathcal{P}$-compact by the generalized Arzela–Ascoli theorem. To show this, we verify the vanishing of the noncompactness measure $\mu(\{(K_n u_n)(t)\}) = 0$ for all $t \in [0, T]$. Let us consider the relation

$$(K_n v_n)(t) = \exp(tA_n)y_n + \psi_n^\tau(t) + \varphi_n^\tau(t),$$

where

$$y_n = \Big(I_n - \exp(TA_n)\Big)^{-1} \int_0^T \exp\big((T-s)A_n\big) f_n\big(s, v_n(s)\big)ds,$$

$$\psi_n^\tau(t) = \exp(\tau A_n)\int_0^{t-\tau} \exp\big((t-s-\tau)A_n\big) f_n\big(s, v_n(s)\big)ds,$$

$$\varphi_n^\tau(t) = \int_{t-\tau}^t \exp\big((t-s)A_n\big) f_n\big(s, v_n(s)\big)ds.$$

By virtue of the boundedness of $\|f_n\big(\cdot, v_n(\cdot)\big)\|_{F_n}$, we can choose the term $\|\varphi_n^\tau(\cdot)\|_{F_n}$ sufficiently small with τ small enough and $\mu(\{\psi_n^\tau\}) = 0$. The sequence $\{y_n\}$ is $\mathcal{P}$-compact.

Step 4. The condition of existence of an isolated uniformly asymptotically stable solution $u(t;x^*)$ of problem (14.19) implies that in a small neighborhood of x^*, say in $S(x^*,\rho) \subset E$, the operator $\mathcal{K}$ is compact, since the set $\mathcal{F}(u_k)(T)$ is compact for any $\{u_k\}, u_k(t) \in S(x^*,\epsilon), t \in [0,T]$, with $\|u_k(0)-x^*\| \le \delta$. The point x^* is an isolated zero of the compact vector field $I-\mathcal{K}$ and $\operatorname{ind}(x^*; I-\mathcal{K})$ is defined. In the same way function $v^*(t) = u(t;x^*), t \in [0,T]$, that is the solution of problem (14.4), is an isolated zero of the field $I-K$ and $\operatorname{ind}(v^*(\cdot); I-K)$ is defined.

From Theorem 14.1.4, it follows that the relation $\operatorname{ind}(x^*; I-\mathcal{K}) = \operatorname{ind}(v^*(\cdot); I-K)$ holds.

Step 5. The condition of uniform asymptotic stability of the solution $u^*(\cdot)$ of problem (14.1) at the point x^* implies that there exists an integer m such that the operator $\mathcal{K}^m$ maps the ball $S(x^*,\delta)$ into itself; more precisely, $\|\mathcal{K}^m(x^*) - \mathcal{K}^m(x)\| \le \phi_{x^*,\delta}(mT) < \delta$ for any $x \in S(x^*,\delta)$. Therefore, this means that $\operatorname{ind}(x^*; I-\mathcal{K}^m) = 1$, and by [45, Theorem 31.1], we obtain $\operatorname{ind}(x^*; I-\mathcal{K}) = 1$. Using Step 4, we have $\operatorname{ind}(v^*(\cdot); I-K) = 1$. Now, $K_n \to K$ compactly, $\operatorname{ind}(v^*(\cdot); I-K) = 1$, and applying the result from [278], we obtain that the set of solutions of problems (14.18) is nonempty, any sequence of solutions $\{v_n^*(\cdot)\}$ is $\mathcal{P}$-compact, and, moreover, $v_n^*(t) \to v^*(t)$ uniformly in $t \in [0,T]$ as $n \to \infty$. The theorem is proved.

14.2. Second Order Equation

In a Banach space E, let us consider the following semilinear Cauchy problem:

$$u''(t) = Au(t) + f(t, u(t), u'(t)), \ \ u(0) = u^0, u'(0) = u^1, \ \ t \in \mathbb{R}, \tag{14.20}$$

with the operator A being a generator of a C_0-cosine operator function and a continuous function f: $\mathbb{R} \times \mathcal{D} \to E$, where $\mathcal{D} \subseteq E^1 \times E$ is a locally closed subset of $E^1 \times E$.

Definition 14.2.1. A *classical solution* of (14.20) on the closed interval $[0,T]$ is a function $u(\cdot) : \mathbb{R} \to E$ such that $u(\cdot)$ is twice continuously differentiable and satisfies (14.20) for all $t \in [0,T]$.

As is known, a classical solution $u(t)$ of problem (14.20) also satisfies the following integral equation (see [274]):

$$u(t) = (Ku)(t) \equiv C(t,A)u^0 + S(t,A)u^1 + \int_0^t S(t-s,A) f(s,u(s),u'(s))ds \tag{14.21}$$

and is a mild solution. Recall that a solution $u(\cdot) \in C^1([0,T];E)$ of Eq. (14.21) is called a *mild solution* of (14.20).

A mild solution of (14.20) is a classical solution if $f(\cdot, u(\cdot))$ is absolutely continuous. Therefore, in general for a continuous function f, a mild solution is not classical.

Existence and uniqueness problems for problem (14.20) were studied, e.g., in [158, 274]. In [158], the case where E is a Banach lattice was also considered.

Assume that $f : J \times E \times E \to E$ satisfies the following conditions:

(C1) $f(\cdot, x, y)$ is strongly measurable for all $x,\ y \in E$, and $f(t, 0, 0) \in L^1(J, E)$;

(C2) for all $x,\ y,\ h,\ k \in E$ and for a.a. $t \in J$,

$$\|g(t, x+h, y+k) - g(t, x, y)\| \leq q(t, \|h\|, \|k\|),$$

where $f : J \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Carathéodory function, $q(t, \cdot, \cdot)$ is nondecreasing for a.a. $t \in J$, the problem

$$u''(t) = M\, f(t, u(t), u'(t)), \qquad u(0) = u_0,\ u'(0) = u_1, \tag{14.22}$$

with some constant M has an upper solution on J for each $(u_0, u_1) \in \mathbb{R}_+^2$, and the zero-function is the only solution of (14.22) when $u_0 = u_1 = 0$.

Theorem 14.2.1 ([158]). *If Conditions* (C1)–(C2) *hold, then for each* $(u^0, u^1) \in E^2 \times E$, *problem* (14.20) *has a unique weak solution* $u(\cdot)$ *on* J*. Moreover,* $u(\cdot)$ *is of the form* $u(t) = u^0 + \int_0^t y(s)\, ds$, $t \in J$, *where* $y(\cdot)$ *is the uniform limit of the sequence* $\{y_n\}_{n=0}^{\infty}$ *of the successive approximations*

$$y_{n+1}(t) = S(t)Au^0 + C(t)u^1 + \int_0^t C(t-s, A) f\left(s, u^0 + \int_0^s y_n(\tau)\right) d\tau, y_n(s))\, ds,$$

with $t \in J,\ n \in \mathbb{N}$, *and with an arbitrarily chosen* $y_0 \in C(J, E)$.

Consider the existence of mild solutions of the problem

$$u''(t) = Au(t) + f(t, u(t)), \qquad u(0) = u^0,\ u'(0) = u^1, \tag{14.23}$$

that lies between assumed upper and lower mild solutions, when E is an ordered Banach space with regular order cone and $f : J \times E \to E$.

Given $u^0,\ u^1 \in E \times E$, we say that $u(\cdot) \in C(J, E)$ is a *lower mild solution* of problem (14.23) on J if

$$u(t) < C(t, A)u^0 + S(t, A)u^1 + \int_0^t S(t-s, A) f(s, u(s))\, ds \tag{14.24}$$

for each $t \in J$. An upper mild solution of (14.23) is defined similarly, by reversing the inequality sign in (14.24). If equality holds in (14.24), we say that $u(\cdot)$ is a mild solution of (14.23).

Let us introduce the following hypotheses on the mappings $f : J \times E \to E$ and $C : J \to B(E)$:

(C3) (14.23) has a lower mild solution $\underline{u}(\cdot)$ and an upper mild solution $\bar{u}(\cdot)$ such that $\underline{u}(\cdot) \leq \bar{u}(\cdot)$, and the functions $f(\cdot, \underline{u}(\cdot))$ and $f(\cdot, \bar{u}(\cdot))$ are Bochner integrable;

(C4) $f(\cdot, u(\cdot))$ is strongly measurable whenever $u(\cdot) \in C(J, E)$;

(C5) $f(t, \cdot)$ is nondecreasing for a.a. $t \in J$;

(C6) $\quad C(t,A)\geq 0$ for all $t\in J$.

If (C6) holds, it follows from (2.8) that $S(t,A)\geq 0$ for each $t\in J$.

Theorem 14.2.2 ([158]). *If Conditions* (C3)–(C6) *hold, then problem* (14.23) *has extremal mild solutions lying between* $\underline{u}(\cdot)$ *and* $\bar{u}(\cdot)$.

Theorem 14.2.3 ([96]). *Assume that Conditions* (A) *and* (B) *hold and the compact resolvents* $R(\lambda;A), R(\lambda;A_n)$ *converge:*

$$R(\lambda;A_n)\to R(\lambda;A)$$

compactly for some $\lambda\in\rho(A)$ *and* $u_n^0\to u^0, u_n^1\to u^1$. *Assume that*

(i) *the functions* f_n, f *are continuous in both arguments and* f *is such that there exists a unique mild solution* $u^*(t)$ *of problem* (14.23) *on* $[0,T]$ *(in this situation, as we will show, ind* $u^*=1$*)*;

(ii) $f_n(t,x_n)\to f(t,x)$ *uniformly in* $t\in[0,T]$ *for* $x_n\to x$;

(iii) *the space* E *is separable.*

Then for almost all n, *the problems*

$$\begin{gathered} u_n''(t)=A_nu_n(t)+f_n(t,u_n(t)),\\ u_n(0)=u_n^0, u_n'(0)=u_n^1, \end{gathered} \tag{14.25}$$

have mild solutions $u_n^*(t), t\in[0,T]$ *in a neighborhood of* $p_nu^*(t)$. *Each sequence* $\{u_n^*(t)\}$ *is* $\mathcal{P}$*-compact and* $u_n^*(t)\to u^*(t)$ *uniformly in* $t\in[0,T]$.

Proof. First, let us prove that the compact convergence of resolvents $R(\lambda;A_n)\to R(\lambda;A)$ is equivalent to the compact convergence of sine operator functions $S_n(t,A_n)\to S(t,A)$ for any $t\geq 0$. Let $\|x_n\|=\mathcal{O}(1)$. We are going to show that from the compact convergence of resolvents $R(\lambda;A_n)\to R(\lambda;A)$ it follows $\mu(\{S_n(t,A_n)x_n\})=0$ for any t, where μ is the measure of noncompactness of sequences. From the identities

$$\begin{aligned} \lambda^2(\lambda^2I-A_n)^{-1}S_n(t,A_n)-S_n(t,A_n)&=\lambda\int_0^\infty e^{-\lambda\eta}C_n(\eta A_n)S_n(t,A_n)d\eta-S_n(t,A_n)\\ &=\frac{1}{2}\lambda\int_0^\infty e^{-\lambda\eta}\big(S_n(t+\eta,A_n)+S_n(t-\eta,A_n)-2S_n(t,A_n)\big)d\eta \end{aligned}$$

we obtain the estimate

$$\begin{aligned} &\|\lambda^2(\lambda^2I-A_n)^{-1}S_n(t,A_n)-S_n(t,A_n)\|\\ &\leq\frac{1}{2}\int_0^\delta e^{-\lambda\eta}\|S_n(t+\eta,A_n)-2S_n(t,A_n)+S_n(t-\eta,A_n)\|d\eta\lambda+\frac{1}{2}\lambda\int_\delta^\infty e^{-\lambda\eta}Me^{\omega\eta}d\eta, \end{aligned}$$

where the first term on the right-hand side is less than ϵ for small δ and the second one is less than ϵ for λ large enough (we recall that if resolvents converge compactly for some λ, then they converge compactly for any λ with sufficiently large $\operatorname{Re}\lambda$). Estimating the measure of noncompactness by

$$\mu(\{S_n(t,A_n)x_n\}) \le \mu(\{\lambda^2(\lambda^2 I_n - A_n)^{-1}S_n(t,A_n)x_n\}) + \|\lambda^2(\lambda^2 I_n - A_n)^{-1}S_n(t,A_n) - S_n(t,A_n)\|$$

we obtain the compact convergence of C_0-sine operator functions. The necessity will be proved if we establish that $\mu(\{(\lambda I_n - A_n)^{-1}x_n\}) = 0$ for $\|x_n\| = O(1)$ under the condition that $S_n(t,A_n) \to S(t,A)$ compactly. We have

$$\mu(\{(\lambda^2 I_n - A_n)^{-1}x_n\}) = \mu\Big(\Big\{\int_0^\infty e^{-\lambda t}S_n(t,A_n)x_n\Big\}\Big)$$
$$\le \mu\Big(\Big\{\int_0^q e^{-\lambda t}S_n(t,A_n)x_n dt\Big\}\Big) + \mu\Big(\Big\{\int_Q^\infty e^{-\lambda t}S_n(t,A_n)x_n dt\Big\}\Big) + \mu\Big(\Big\{\int_q^Q e^{-\lambda t}S_n(t,A_n)x_n dt\Big\}\Big).$$

If q is small enough and Q is large enough, then the first and second terms become less than ϵ. The third term is equal to zero by the uniform continuity of $S_n(\cdot,A_n)$ on $[q,Q]$.

Now we are going to prove that the compact convergence of C_0-sine operator functions and condition (ii) imply that $K_n \to K$ compactly. It is clear that $K_n \to K$. Let $\{u_n\}$ be a sequence of functions $u_n \in C(0,T;E_n)$ such that $\|u_n\|_{C(0,T;E_n)} = O(1)$ as $n \to \infty$. To prove that $\{K_n u_n\}$ is compact, we apply the theorem from [46]. The sequence of functions $\{K_n u_n\}, K_n u_n \in C(0,T;E_n)$, is uniformly bounded, equicontinuous, and for any $t \in [0,T]$, the operator K_n maps the bounded set of functions $\{u_n\}$ into a precompact set. Therefore, $K_n \to K$ compactly. Now, from [278], it follows that $\gamma(I-K;\partial S_r) = \gamma(I_n - K_n;\partial S_{n,r})$ as $n \ge n_0$. If we establish that $\gamma(I-K;\partial S_r) \ne 0$, then by Theorem 3 in [278], it follows that solutions of (14.25) do exist in a neighborhood of $p_n u^*(t)$, each sequence $\{u_n^*(t)\}$ is $\mathcal{P}$-compact, and $u_n^*(t) \to u^*(t)$ uniformly with respect to $t \in [0,T]$; this will prove the theorem.

Therefore, let us show that $\gamma(I-K;\partial S_r) = 1$. It follows from the assumption of the theorem that the operator K has no fixed points on the boundary ∂S_r, where $S_r = \{u : \|u - u^*\| < r\}$. We want to show that for the operator $(Ku)(t) = C(t)u^0 + S(t,A)u^1 + \int_0^t S(t-s,A)f(s,u(s))ds$, with a continuous function f, the index of the fixed point u^* is equal to $\gamma(I-K;\partial S_r) = 1$. To do this, we define the operator

$$G_\lambda u = K(P_\lambda u) + u^* - K(P_\lambda u^*),$$

where $K(u^*) = u^*$ and the operator P_λ is defined by the formulas

$$(P_\lambda u)(t) = u(t-\lambda) \text{ for } \lambda < t \le T,$$

$$(P_\lambda u)(t) = u(0)) \text{ for } t \in [0,\lambda].$$

We complete the proof if we prove the following two assertions:

$$\gamma(I - G_\lambda; \partial S_r) = \gamma(\Phi_1; \partial S_r) = 1,$$

where $\Phi_1(u) = u - u^*$, and

$$\gamma(I - G_\lambda; \partial S_r) = \gamma(I - G_0; \partial S_r) = \gamma(I - K; \partial S_r)$$

for sufficiently small λ. The fields $\Phi_1(u) = u - u^*$ and $\Phi_2(u) = u - G_\lambda u$ are connected by a linear compact nondegenerate deformation (see [45, Sec. 19.1]):

$$H(\mu, \lambda)u = u - \mu G_\lambda u - (1 - \mu)u^*, \qquad 0 \le \mu \le 1,$$

i.e., Φ_1 and Φ_2 are linearly homotopic. The operator H has no singular points on ∂S_r. To prove this, we assume the contrary: there exist $v^* \neq u^*$ and $H(\mu, \lambda)v^* = 0$. Then by the formula

$$v^* = \mu K(P_\lambda v^*) - \mu K(P_\lambda u^*) + u^*,$$

we first obtain $v^*(0) = u^*(0)$, and, therefore, because of the relation $(P_\lambda v^*)(t) = (P_\lambda u^*)(t)$ for $0 \le t \le \lambda$, we have

$$v^*(t) = u^*(t) \text{ for } t \in [0, \lambda]. \tag{14.26}$$

Repeating the same arguments, we obtain

$$v^*(t) = u^*(t) \text{ for } t \in [0, 2\lambda],$$

since $(P_\lambda v^*)(t) = (P_\lambda u^*)(t)$ for $0 \le t \le 2\lambda$ by virtue of (14.26). In this way, we can arrive at 3λ and so on; this means that $u^* = v^*$. Since the operator H has no singular points on ∂S_r, H is a linear compact deformation. Clearly,

$$\gamma(I - H(0, \lambda); \partial S_r) = \gamma(\Phi_1; \partial S_r) = 1,$$

and, therefore, by [45, Theorem 20.1],

$$1 = \gamma(I - H(1, \lambda); \partial S_r) = \gamma(I - G_\lambda; \partial S_r).$$

The operator G_λ is compact for any λ (see [284]), and, moreover, $\{\cup_{\lambda \in [0,T]} G_\lambda u : u \in S_r\}$ is precompact (if this set is not relatively compact, then there exist two sequences $\{\lambda_k\}$ and $\{u_k\}$ such that $\{G_{\lambda_k} u_k\}$ is not compact, which contradicts the compactness of K). Clearly, $P_\lambda \to P_{\lambda_0}$ strongly in $C([0, T]; E)$ as $\lambda \to \lambda_0$. Let $v_k \to v_0$, and let $\lambda_k \to \lambda_0$. Since

$$\begin{aligned} G_{\lambda_k} v_k - G_{\lambda_0} v_0 &= K(P_{\lambda_k} v_k) + u^* - K(P_{\lambda_k} u^*) - K(P_{\lambda_0} v_0) - u^* + K(P_{\lambda_0} u^*) \\ &= K(P_{\lambda_k} v_k) - K(P_{\lambda_0} v_0) + K(P_{\lambda_0} u^*) - K(P_{\lambda_k} u^*) \to 0 \text{ as } \lambda_k \to \lambda_0, \end{aligned}$$

we have that the operator G_λ is continuous in both arguments, and, as we have seen, the set $\{y : y = G_\lambda u, \|u - u^*\| \le r, 0 \le \lambda \le T\}$ is relatively compact. Since $G_\lambda \to G_0$ compactly as $\lambda \to 0$, by [278], the operators G_λ have no fixed points on ∂S_r for small λ, and for the same λ, we obtain

$$\gamma(I - G_\lambda; \partial S_r) = \gamma(I - G_0; \partial S_r) = \gamma(I - K; \partial S_r).$$

The theorem is proved.

Acknowledgments. S. Piskarev acknowledges the support of Russian Foundation for Basic Research (01-01-00398), DAAD of Germany and Chinese Academy of Sciences and Control System Laboratory of Academy of Mathematics and System Sciences, Academia Sinica.

REFERENCES

1. A. Ashyralyev, *Well-Posedness of the Elliptic Equations in a Space of Smooth Functions. Boundary Value Problems For Nonclassical Equations of Mathematical Physics* [in Russian], Sibirsk. Otdel, SSSR-Novosibirsk, **2**, No. 2, 82–86 (1989).
2. A. Ashyralyev, "Well-posed solvability of the boundary value problem for difference equations of elliptic type," *Nonlinear Anal. Theory, Meth. Appl.,* **24**, 251–256 (1995).
3. A. Ashiraliev, "On the coercive solvability of parabolic equations in spaces of smooth functions," *Izv. Akad. Nauk TurkSSSR, Ser. Fiz.-Mat., Khim., and Geol. Nauk*, **3**, 3–13 (1989).
4. G. Babescu, "On a certain connection between second-order linear differential and cosine operator functions on Banach spaces," *Lucr. Semin. Mat. Fiz. Inst. Politehn. Trian Vuia, Timisoara*, **146**, 13–16 (1982).
5. G. Babescu, "Spectral theorems for functions with values of cosine and sine operators," *Bull. Sci. Tehn. Inst. Politehn. Timisoara. Mat. Fiz.*, **27**, No. 1, 13–20 (1982).
6. A. G. Baskakov, "Harmonic analysis of cosine and exponential operator functions," *Mat. Sb.*, **124(166)**, No. 1(5), 68–95 (1984).
7. A. G. Baskakov, *Harmonic Analysis of Linear Operators* [in Russian], Voronezh Univ., Voronezh (1987).
8. O. V. Besov, V. P. Il'in, and S. M. Nikol'skii, *Integral Representations of Functions and Embedding Theorems* [in Russian], Nauka, Moscow (1996).
9. L. Berg and I. Lëfstrem, *Interpolation Spaces. An Introduction* [Russian translation], Mir, Moscow (1980).
10. G. M. Vainikko and M. Shlapikiene, "To a certain S. G. Krein theorem on perturbation of operators generating analytic semigroups," *Uch. Zap. Tartu Univ.*, **281**, 186–189 (1971).

11. V. V. Vasil'ev, *Introduction to the Theory of Abstract Differential Equations* [in Russian], BGU, Belgorod (1988).
12. V. V. Vasil'ev, "On the coercive solvability of abstract parabolic equations with a constant operator," *Differents. Uravn.,* **14**, No. 8, 1507–1510 (1978).
13. V. V. Vasil'ev and E. V. Emel'yanova, "The coercive solvability of abstract parabolic equations in weighted spaces," *Deposit at VINITI*, No. 6621-V88 (1988), 34 p.
14. V. V. Vasil'ev, "On the set of equations of thermoelastisity," *Deposit. at VINITI*, No. 1681-V87 (1987), 14 p.
15. V. V. Vasil'ev, "Spectral properties of a certain operator matrix," *Deposit at VINITI*, No. 1573-V88 (1988), 9 p.
16. V. V. Vasil'ev, S. G. Krein, and S. I. Piskarev, "Operator semigroups, cosine operator functions, and linear differential equations," In: *Progress in Science and Technology, Series on Mathematical Analysis* [in Russian], Vol. 28, All-Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1990), pp. 87–202.
17. V. V. Vasil'ev and S. I. Piskarev, *Differential Equations in a Banach Space, Theory of Operator Semigroups* [in Russian], MGU, Moscow (1996).
18. V. V. Vasil'ev and S. I. Piskarev, *Bibliographical Index, Differential Equations in Abstract Spaces*, NIVTs MGU, *Electronic Journal*,
http:// www.srcc.msu.su/num_anal/list_wrk/page_5u.html
PostScript (1115.2 Kb), PostScript.zip (336.1 Kb), Moscow (2001).
19. V. S. Vladimirov, Yu. N. Drozhzhinov, and B. I. Zav'yalov, *Tauberian Theorems of Distributions* [in Russian], Nauka, Moscow (1986).
20. J. Goldstein, *Semigroups of Linear Operators and Their Applications* [Russian translation], Vishcha Shkola, Kiev (1989).
21. M. L. Gorbachuk and I. T. Matsishin, "On solutions of evolutionary equations with degeneration in a Banach space," In: *Spectral Theory of Operator-Differential Equations* [in Russian], Kiev (1986), pp. 5–10.
22. R. Ya. Grabovskaya and V. I. Kononenko, "On the cosine operator function generated by the sum of two commuting operators," *Deposit. at VINITI*, No. 783-84 (1984), 6 p.
23. R. Ya. Grabovskaya and V. I. Kononenko, "On the representation of a cosine function whose generator is the sum of two commuting operators," In: *VII School on Operator Theory in Function Spaces* [in Russian], Part 1, Riga (1983), pp. 63–64.

24. R. Ya. Grabovskaya, V. I. Kononenko, and S. D. Shmulevich, "On the connection between solutions of certain linear differential equations in Banach spaces," *Deposit. at VINITI*, No. 2479-85 (1985), 13 p.
25. V. P. Gudkin, D. A. Dyment, and V. A. Matveev, "Coercive solvability of abstract parabolic equations in Hölder spaces with weight," In: *Proc. KhABIZhT Conf.* [in Russian]
26. I. N. Gurova, "On a certain topological method for studying difference schemes," *Dokl. Akad. Nauk SSSR*, **248**, No. 1, 25–28 (1979).
27. Yu. L. Daletskii and M. G. Krein, *Stability of solutions of differential equations in a Banach space* [in Russian], Nauka, Moscow (1970).
28. Yu. N. Drozhzhinov and B. I. Zav'yalov, "Quasiasymptotics of distributions and Tauberian theorems in complex domain," *Mat. Sb.*, **102(144)**, No. 3, 372–390 (1977).
29. Yu. V. Egorov, *Linear Differential Equations of Principal Type* [in Russian], Nauka, Moscow (1984).
30. V. V. Ivanov, I. V. Mel'nikova, and F. M. Filinkov, *Differential-Operator Equations and Ill-Posed problems* [in Russian], Nauka, Moscow (1995).
31. T. Kato, *Perturbation Theory of Linear Operators* [Russian translation], Mir, Moscow (1972).
32. P. A. Kirichuk, "On perturbation of an operator function of the Mittag-Leffler type," *Dokl. Akad. Nauk USSR*, No. 11, 9–12 (1990).
33. Ph. Clement et al. (Eds.), *One-Parameter Semigroups* [Russian translation], Mir, Moscow (1992).
34. V. A. Kostin, "Abstract strongly continuous pairs of trigonometrical transformation groups," *Deposit. at VINITI*, No. 829-80 (1980), 19 p.
35. V. A. Kostin, "To the Cauchy problem for ordinary differential equations in a Banach space," In: *Partial Differential Equations* [in Russian], Vol. 93, Novosibirsk (1989), pp. 93–116.
36. V. A. Kostin, "On the exact uniformly well-posed solvability of the Cauchy problem," *Dokl. Akad. Nauk SSSR*, **319**, No. 1, 38–41 (1991).
37. V. A. Kostin, "To the solution of a certain problem related to the abstract cosine function," *Dokl. Ross. Akad. Nauk*, **336**, No. 5 (1994).
38. V. A. Kostin, "On the Cauchy problem for an abstract hyperbolic equation," In: *Int. Sci. Conf. on Differ. and Integral. Equat, Abstracts of Reports*, Samara (1995), p. 135.
39. V. A. Kostin. "To the Cauchy problem for abstract differential equations with fractional derivatives," *Dokl. Ross. Akad. Nauk*, **326**, No. 4, 597–600 (1992).
40. V. A. Kostin, "On analytic semigroups and strongly continuous cosine functions," *Dokl. Akad. Nauk SSSR*, **307**, No. 4, 796–799 (1989).

41. V. A. Kostin, "On a certain criterion of a strongly continuous cosine function," In: *XIV School on Operator Theory, Abstracts of Reports* [in Russian], Novgorod (1989), p. 32.
42. V. A. Kostin, "On a certain representation of a strongly continuous cosine function," *Deposit. at VINITI*, No. 2959-83 (1983), 14 p.
43. V. A. Kostin, "To the Solomyak–Iosida theorem on analytic semigroups," *Algebra Anal.*, **11**, No. 1, 91–106 (1999).
44. A. N. Kochubei, "Cauchy problem for evolutionary equation of fractional order," *Differents. Uravn.*, **25**, No. 8, 1359–1368 (1989).
45. M. A. Krasnosel'skii and P. P. Zabreiko, *Geometric Methods in Nonlinear Analysis* [in Russian], Nauka, Moscow (1975).
46. M. A. Krasnosel'skii et al., *Integral Operators in Spaces of Lebesgue Integrable Functions* [in Russian], Nauka, Moscow (1966).
47. S. G. Krein, *Linear Differential Equations in a Banach Space* [in Russian], Nauka, Moscow (1967).
48. S. G. Krein and M. I. Khazan, "Differential equations in a Banach space," In: *Progress in Science and Technology, Series on Mathematical Analysis* [in Russian], Vol. 21, All-Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1983), pp. 130–264.
49. S. G. Krein, Yu. M. Petunin, and E. I. Semenov, *Interpolation of Linear Operators* [in Russian], Nauka, Moscow (1978).
50. I. V. Mel'nikova, "Well-posedness of the Cauchy problem for a second-order equation, and properties of the generalized resolvent," *Deposit. at VINITI*, No. 5523-82 (1982), 10 p.
51. I. V. Melnikova, "On the Cauchy problem for second-order equations," *Differents. Uravn.,* **19**, No. 3, 537–538 (1983).
52. I. V. Mel'nikova, "Operator functions $E(t)$ and $ES(t)$, and the well-posedness of the Cauchy problem for second-order equations," *Deposit. at VINITI*, No. 4537-82 (1982), 14 p.
53. I. V. Melnikova, "Family M, N of operator functions and second-order equations in a Banach space," *Izv. Vuzov, Mat.*, No. 2, 45–52 (1985).
54. I. V. Mel'nikova, "Myadera–Feller–Philips-type theorem for a complete second-order equation in a Banach space," *Izv. Vuzov, Mat.*, No. 4, 34–40 (1985).
55. I. V. Mel'nikova and A. I. Filinkov, "Classification and well-posedness of the Cauchy problem for second-order equations in a Banach space," *Dokl. Akad. Nauk SSSR*, **276**, No. 5, 1066–1071 (1984).
56. I. V. Mel'nikova and A. I. Filinkov, "On the principal with respect to the Cauchy problem operator for a complete second-order equation in a Banach space," *Deposit. at VINITI*, No. 3719-V86 (1986), 19 p.

57. I. V. Mel'nikova and A. I. Filinkov, "Connection of the well-posedness of the Cauchy problem for an equation and a system in a Banach space," *Dokl. Akad. Nauk SSSR*, **300**, No. 2, 280–284 (1988).

58. I. P. Natanson, *Theory of Functions of a Real Variable* [in Russian], Nauka, Moscow (1974).

59. Yu. M. Petunin and A. N. Plichko, *Theory of Characteristics of Spaces and Its Applications* [in Russian], Vysha Shkola, Kiev (1980).

60. S. Piskarev, "Convergence of difference schemes for solving nonlinear parabolic equations," *Mat. Zametki*, **44**, No. 1, 112–123 (1988).

61. S. I. Piskarev, "Compact convergence in approximation of differential equations in a Banach space," *Uch. Zap. Tartu. Univ.,* **633**, 11–18 (1982).

62. S. I. Piskarev, "Periodic and almost-periodic cosine operator functions," *Mat. Sb.,* **118,** No. 3, 386–398 (1982).

63. S. I. Piskarev, "Almost-periodic solutions of second-order equations," *Sib. Mat. Zh.* **25**, No. 3, 137–147 (1984).

64. S. I. Piskarev, "Compactness properties in the theory of cosine operator functions," *Mat. Zametki*, **51**, No. 5, 151–153 (1992).

65. E. A. Polyanskii, *Method for Correction of a Solution of a Parabolic Equation in an Inhomogeneous Wave Guide* [in Russian], Nauka, Moscow (1985).

66. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. 1, Functional Analysis* [Russian translation], Mir, Moscow (1977).

67. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. 2, Harmonic Analysis, Self-Adjointness* [Russian translation], Mir, Moscow (1978).

68. T. I. Smagina, "On a certain theorem of Phillips," In: *Operator Equations in Function Spaces* [in Russian], Voronezh (1986), pp. 91–93.

69. V. I. Sobolev, *Lectures on Additional Chapters of Mathematical Analysis* [in Russian], Nauka, Moscow (1968).

70. P. E. Sobolevskii, "Coercive inequalities for abstract parabolic equations," *Dokl. Akad. Nauk SSSR*, **157**, No. 1 (1964).

71. P. E. Sobolevskii, "The coercive solvability of difference equations," *Dokl. Akad. Nauk SSSR,* **201**, 1063–1066 (1971).

72. P. E. Sobolevskii, "Semigroup theory and stability of difference schemes," In: *Operator Theory on Functional Spaces* [in Russian], Novosibirsk (1977), pp. 304–337.

73. H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators* [Russian translation], Mir, Moscow (1980).

74. D. Henry, *Geometric Theory of Semilinear Parabolic Equations* [Russian translation], Mir, Moscow (1985).

75. P. Hess, *Periodic-Parabolic Boundary-Value Problems and Positivity* [Russian translation], Mir, Moscow (2001).

76. E. Hille and P. Phillips, *Functional Analysis and Semigroups* [Russian translation], IL, Moscow (1962).

77. H. Amann, "Linear and quasilinear parabolic problems. Volume 1. Abstract linear theory," *Monogr. Math.,* Vol. 87, Birkhäuser Verlag (1995).

78. M. Amerio and G. Prouse, *Almost-Periodic Functions and Functional Equations,* New York (1971).

79. S. B. Agase and V. Raghavendra, "Existence of mild solutions of semilinear differential equations in Banach spaces," *Indiana J. Pure Appl. Math.,* **21**, No. 9, 813–821 (1990).

80. W. Arendt, O. El-Mennaoui, and V. Kéyantuo, "Local integrated semigroups: evolution with jumps of regularity," *J. Math. Anal. Appl.*, **186**, No. 2, 572–595 (1994).

81. W. Arendt and J. Pruss, "Vector valued Tauberian theorems and asymptotic behaviour of linear Volterra equations," *SIAM Appl. Math.,* **23**, 412–448 (1992).

82. A. Ashyralyev and P. E. Sobolevskii, *Well-Posedness of Parabolic Difference Equations,* Birkhäuser Verlag (1994).

83. A. Ashyralyev, S. Piskarev, and L. Weis, "On well-posedness of difference schemes for abstract parabolic equations in $L^p([0,T];E)$ spaces," to appear in *Numer. Funct. Anal. and Optimiz.*

84. Pascal Auscher, Alan McIntosh, and Andrea Nahmod, "The square root problem of Kato in one dimension, and first order elliptic systems," *Indiana Univ. Math. J.*, **46**, No. 3, 659–695 (1997).

85. L. Autret, "Entire vectors and time reversible Cauchy problems," In: *Semigroup Forum,* **46**, 347–351 (1993).

86. Gh. Babesku, "Bromwich's type representations for cosine and sine functions," *Semin. Equat. Funct. Univ. Timisoara,* **85**, 1–7 (1987).

87. Gh. Babesku, "Regularity and uniform continuity properties of cosine and sine class of operators," In: *Lucr. Semin. Math. Fis. Inst. Politehn. Timisoara* (1983), pp. 47–50.

88. Gh. Babesku, "Über Kosinus Familien von liniaren stark stetigen Operatoren und Ihren infinitesimalen Generatoren," *Bul. Sti. Tehn. Inst. Politehn. Timisoara. Ser. Math. Fiz.,* **25**, No. 1, 12–17 (1980).

89. J. B. Baillon, "Caracter borne de certains generatours de semi-groupes lineaires dans les espaces de Banach," *C. R. Acad. Sci. Paris,* **290**, 757–760 (1980).

90. A. V. Balakrishnan, "An operational calculus for infinitesimal generators of semigroups," *Trans. Amer. Math. Soc.,* **91**, 330–353 (1959).

91. L. A. Bales, "Higher order single step fully discrete approximations for second order hyperbolic equations with time dependent coefficients," *SIAM J. Numer. Anal.*, **23**, No. 1, 27–43 (1986).

92. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spases,* Noordhoff, Leyden (1976).

93. H. Bart, "Periodic strongly continuous semigroups," *Ann. Mat. Pura ed Appl.*, **115**, 311–318 (1977).

94. H. Bart and S. Golberg, "Characterization of almost periodic strongly continuous groups and semigroups," *Math. Ann.,* **236**, No. 2, 105–116 (1978).

95. Ph. Benilan, M. Crandall, and A. Pazy, "Nonlinear evolution equations in Banach spaces" (in press).

96. N. A. Bobylev et al., "Semilinear approximation of semilinear periodic problems in Banach spaces," *Nonlinear Analisys,* **33**, 473–482 (1988).

97. H. Brezis, *Operateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert,* North-Holland, Amsterdam–London (1973).

98. P. L. Butzer and H. Berens, *Semigroups of Operators and Approximation,* New York (1967).

99. R. Chander and A. B. Buche, "On some analytic properties of the modified sine operator," *Indian J. Pure Appl. Math.*, **12**, 585–593 (1981).

100. D.-K. Chyan, S.-Y. Shaw, and S. Piskarev, "On maximal regularity and semivariation of cosine operator functions," *J. London Math. Soc. (2)*, **59**, No. 3, 1023–1032 (1999).

101. J.-Ch. Chang and S.-Y. Shaw, "Rates of approximation and egrodic limits of resolvent families," *Arch. Math.,* **66,** 320–330 (1996).

102. R. Chander and H. Singh, "On the measurability and continuity properties of modified exponential cosine operator," *J. Indiana Math. Soc.,* **44,** 365–368 (1980).

103. I. Cioranescu, "Characterizations of almost periodic strongly continuous cosine operator functions," *J. Math. Anal.,* **57**, No. 1, 222–229 (1986).

104. I. Cioranescu and V. Keyantuo, "On operator cosine functions in UMD spaces," *Semigroup Forum,* **63**, 429–440 (2001).

105. I. Cioranescu and C. Lizama, "Spectral properties of cosine operator functions," *Aequat. Math.,* **36**, No. 1, 80–98 (1988).

106. I. Cioranescu and C. Lizama, "Some applications of Fejer's theorem to operator cosine functions in Banach spaces," *Proc. Amer. Math. Soc.,* **125**, No. 8, 2353–2362 (1997).

107. I. Cioranescu and U. Neumann, "Stielties vectors and cosine functions generators," *Stud. math. (PPL),* **1**, 1–7 (1987).

108. I. Cioranescu and P. Ubilla, "A characterization of uniformly bounded cosine functions generators," *Integral Equat. Oper. Theory,* **12**, No. 1, 1–11 (1989).
109. W. Choinacki, "Group representations of bounded cosine functions," *J. Reine Angew. Math.,* **478**, 61–84 (1996).
110. M. G. Crandall, "A remark on semilinear perturbations of abstract parabolic equations," *Nonlinear Anal., Theory, Meth. and Appl.,* **9**, No. 12, 1331–1335 (1985).
111. J. R. Cuthbert, "On semigroups such that $T(t) - I$ is compact for some $t > 0$," *Z. Warscheinlichkeitstheorie,* **18**, Nos. 1–2, 9–16 (1971).
112. G. Da Prato, "Abstract differential equations, maximal regularity and linearization," *Proc. Symp. Pure Math.,* **45**, No. 1, 359–370 (1986).
113. G. Da Prato and E. Giusti, "A characterization on generators of abstract cosine functions," *Boll. Del. Unione Mat.,* **22**, 367–362 (1967).
114. G. Da Prato and M. Ianelli, "Linear integrodifferential equations in Banach spaces," *Rend. Semin. Math. Padova,* **62**, 207–219 (1980).
115. D. Daners and P. K. Medina, "Abstract evolution equations. Periodic problems and applications," In: *Pitman Res. Notes in Math.*, **279**, Longman, New York (1992).
116. R. de Laubenfels, Z. Huang, S. Wang, and Y. Wang, "Laplace transforms of polynomially bounded vector-valued functions and semigroups of operators," *Israel J. Math.*, **98**, 189–207 (1997).
117. R. de Laubenfels, *Existence Families, Functional Calculi and Evolution Equations, Lecture Notes in Math.*, **1570**, Springer-Verlag (1994).
118. R. de Laubenfels, "Integrated semigroups and integro-differential equations," *Math. Z.*, **204**, 501–514 (1990).
119. R. de Laubenfels and S. Piskarev, "The growth rate of cosine families," *J. Math. Anal. Appl.,* **196**, 442–451 (1995).
120. W. Desch and W. Schappacher, "Some generation results for perturbed semigroups," In: *Trends in Semigroup Theory and Applications,* Ph. Clement et al. (Eds.), Marcel Dekker, New York (1989), pp. 125–152.
121. J. W. Dettman, "Related semigroups and the abstract Hilbert transform," *Appl. Anal.*, **79A**, 173–182 (1977).
122. J. W. Dettman, "Saturation theorems connected with the abstract wave equation," *SIAM J. Math. Anal.,* **9**, No. 1, 59–64 (1978).
123. O. Diekmann, M. Gyllenberg, and H. Thieme, "Pertubing semigroups by solving Stieltjes renewal equations," *Differ. Integral Equat.*, **6**, 155–181 (1993).

124. B. Eberhard and G. Greiner, "Baillon's theorem on maximal regularity," *Acta Appl. Math.,* **27,** 47–54 (1992).

125. O. El-Mennaoni, *Traces de Semi-Groupes Holomorphes Singuliers l'Origine et Comportement Asymptotique,* Thise, Besançon (1992).

126. O. El-Mennaoni and V. Keyantuo, "Trace theorems for holomorphic semigroups and the second order Cauchy problems," *Proc. Amer. Math. Soc.,* **124**, No. 5, 1445–1458 (1996).

127. E. V. Emel'anova and V. Vasil'ev, "The coercive solvability of abstract parabolic equations in Banach spaces," In: *Lecture Notes in Pure and Appl. Math.*, **155**, 167–171 (1993).

128. K.-J. Engel, *Operator Matrices and Systems of Evolution Equations* (1999).

129. K.-J. Engel and R. Nagel, *One Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Math.*, Springer, Berlin (2000).

130. T. Ergens, B. Karasözen, and S. Piskarev, "Approximation for semilinear Cauchy problems involving second order equations in separable Banach spaces," *Nonlinear. Anal.,* **28**, 1157–1165 (1997).

131. H. O. Fattorini, "A note of fractional derivatives of semigroups and cosine functions," *Pacif. J. Math.,* **109**, No. 2, 335–347 (1983).

132. H. O. Fattorini, "On the growth of solutions to second order differential equations in Banach spaces," *Proc. Roy. Soc. Edinburgh,* **101A**, Nos. 3–4, 237–252 (1985).

133. H. O. Fattorini, "Ordinary differential equations in linear topological spaces, I," *J. Differential Equations,* **5**, No. 1, 72–105 (1969).

134. H. O. Fattorini, "Ordinary differential equations in linear topological spaces, II," *J. Differential Equations,* **6**, 50–70 (1969).

135. H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces,* North Holland, Amsterdam (1985).

136. H. O. Fattorini, "Two points boundary value problems for operational differential equations," In: *Ann. Sc. Norm. Super.,* Ser. IV, Vol. I, Nos. 1–2, pp. 63–79 (1974).

137. H. O. Fattorini, "The underdetermined Cauchy problem in Banach spaces," *Math. Ann.,* **200**, 103–112 (1973).

138. H. O. Fattorini, "Un teorema de perturbacion para generatores de functiones coseno," *Revista de la Union Mathematica Argentina,* **25**, 199–211 (1971).

139. H. O. Fattorini, "Uniformly bounded cosine functions in Hilbert spaces," *Indiana Univ. Math. J.,* **20**, 411–425 (1970).

140. H. O. Fattorini and A. Radnitz, "The Cauchy problem with incomplete initial data in Banach spaces," *Mich. Math. J.,* **18**, 291–320 (1971).

141. E. Giusti, "Funzioni coseno periodische," *Boll. Unione Mat. Ital.,* **22**, 478–485 (1967).

142. J. A. Goldstein, "On the connection between first and second order differential equations in Banach spaces," *J. Math. Anal. Appl.,* **30**, No. 2, 246–251 (1970).

143. J. A. Goldstein, "Semigroups and second-order differential equations," *J. Funct. Anal.,* **4**, 50–70 (1969).

144. J. A. Goldstein, "The universal addability problem for generators of cosine functions and operator groups," *Houston J. Math.,* **6**, No. 3, 365–373 (1980).

145. J. A. Goldstein and S. Oharu, "A remark on nonlinear cosine functions," *Semigroup Forum*, **34**, No. 3, 359–366 (1987).

146. J. A. Goldstein, C. Radin, and R. E. Schowalter, "Convergence rates of ergodic limits for semigroups and cosine functions," *Semigroup Forum,* **16**, 89–95 (1978).

147. W. T. Gowers and B. Maurey, "Banach spaces with small spaces of operators," *Math. Ann.*, **307**, 543–568 (1997).

148. H. Grabmüller, "Cosine families of unbounded operators," *Appl. Anal.,* **19**, 1–38 (1985).

149. H. Grabmüller, "Cosinus-Familien unbeschrankter Operatoren," *ZAMM Angew. Math. Mech.,* **65**, No. 5, 338–340 (1985).

150. R. Grimmer and J. Prüss, "On linear Volterra equations in Banach spaces," *Comp. Math. Appl.,* **11**, 189–205 (1985).

151. B. Z. Guo, "On the exponential stability of C_0-semigroups on Banach spaces with compact perturbations," *Semigroup Forum,* **59**, No. 2, 190–196 (1999).

152. D. Guidetti, "Strongly continuous semigroups, weak solutions and the variation of parameter formula in locally convex space," *Boll. Unione Mat. Ital.,* **C4**, No. 1, 431–440 (1985).

153. D. Guidetti. "The parabolic mixed Cauchy–Dirichlet problem in spaces of functions which are Hölder continuous with respect to space variables," *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **7**, No. 3, 161–168 (1996).

154. D. Guidetti and S. Piskarev, "On maximal regularity of difference schemes for parabolic equations in $C^\theta(\overline{\Omega})$ spaces," *Preprint* No. 8, Dipartimento di Matematica, Universitaá degli Studi di Bologna (1997).

155. D. Guidetti and S. Piskarev, "On maximal regularity of Crank–Nicholson schemes for parabolic equations in $C^\theta(\overline{\Omega})$ space," In: *Proc. Int. Conf. 'Function spaces. Differential operators. Problems in mathematical education'*, Vol. 2, pp. 2231–2237, RUDN, Moscow (1998).

156. D. Guidetti and S. Piskarev, "Stability of Rothe's scheme and maximal regularity for parabolic equations in $C^\theta(\overline{\omega})$," In: *Progress in partial differential equations*, Vol. 1 (Pont-à-Mousson, 1997), pp. 167–180. Longman, Harlow (1998).

157. K. Gustavson and G. Lumer, "Multiplicative pertubations of semigroup generators," *Pacif. J. Math.,* **41**, No. 3, 731–742 (1972).

158. S. Heikkila and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations,* Marcel Dekker, New York–Basel (1994).

159. H. R. Henriques, "Cosine operator families such that $C(t) - I$ is compact for all $t > 0$," *Indian J. Pure and Appl. Math.,* **16**, No. 2, 143–152 (1985).

160. H. R. Henriquez, "On Stepanov-almost periodic semigroups and cosine functions of operators," *J. Math. Anal. Appl.,* **146**, No. 2, 420–433 (1990).

161. R. Hersh, "Explicit solutions of a class of higher order abstract Cauchy problem," *J. Differ. Equations,* **8**, No. 3, 570–579 (1970).

162. M. Hieber, "Integrated semigroups and differential operators on $L^p(\mathbb{R}^N)$-spaces," *Math. Ann.*, **291**, 1–16 (1991).

163. M. Hieber and J. Prüss, "Heat cernels and maximal $L^p - L^q$ estimates for parabolic evolution equations," *Preprint* (1998).

164. C. S. Honig, *Volterra–Stieltjes Integral Equations,* North-Holland, Amsterdam (1975).

165. R. H. W. Hoppe, "Discrete approximations of cosine operator functions, I," *SIAM J. Numer. Anal.,* **19**, No. 6, 1110–1128 (1982).

166. R. H. W. Hoppe, "Interpolation of cosine operator functions," *Ann. Mat. Pura Appl. (4),* **136**, 183–212 (1984).

167. L. Hörmander, "Estimates for translation invariant operators in L^p spaces," *Acta Math.,* **104**, 93–140 (1960).

168. B. Jefferies and S. Piskarev, "Tauberian theorems for C_0-semigroups," In: *Proc. 4th International Conference on Functional Analysis and Approximation Theory, Vol.* 2, Maratea, Italy (2000). Supplemento di rendicoti del Circolo mathematico di Palermo, Ser. II, No. 68 (2002), pp. 513–521.

169. B. Jefferies and S. Piskarev, "Tauberian theorems for cosine operator functions," *Proc. V.A. Steklov Institute,* **236**, 474–460 (2002).

170. Jung Chan Chang and S.-Y. Shaw, "Perturbation theory of abstract Cauchy problems and Volterra equations," *Nonlinear Anal. Theory Meth. Appl.,* **30**, No. 6, 3521–3528 (1997).

171. N. J Kalton and G. Lancien, "A solution to the problem of L^p-maximal regularity," *Math. Z.*, **235**, No. 3, 559–568 (2000).

172. N. Kalton and L. Weis, "The H^∞ calculus and sums of closed operators" (*in press*).
173. S. Kantorovitz, "Analytic families of semigroups," *Semigroup Forum,* **54**, 356–363 (1997).
174. S. Kantorovitz, "C^n-operational calculus, non-commutative Teylor formula and perturbation of semigroups," *J. Funct. Anal.,* **113**, No. 1, 139–152 (1993).
175. S. Kantorovitz, "Semigroups of operators and spectral theory," *Pitman Reseach Notes in Math. Series,* **330**, Longman (1995), 135 p.
176. S. Kantorovitz and S. Piskarev, "Mean stability of semigroups and cosine operator functions," to appear in *Taiwan. J. Math.*
177. V. Keyantuo, "Integrated semigroups and related partial differential equations," *J. Math. Anal. Appl.,* **212**, 135–153 (1997).
178. V. Keyantuo, "The Laplace transform and the ascent method for abstract wave equations," *J. Differ. Equations,* **122**, 27–47 (1995).
179. J. Kisynski, "On cosine operator functions and one parameter groups of operators," *Studia Math.,* **44**, 93–105 (1972).
180. V. Kozlov and V. Maz'ya, *Differential Equations with Operator Coefficients with Applications to Boundary Value Problems for Partial Differential Equations,* Springer Monogr. in Math. (1999), XX.
181. S. Kurepa, "A cosine functional equation in Hilbert space," *Can. J. Math.,* **12**, 45–49 (1960).
182. S. Kurepa, "A cosine functional eqation in n-dimensional vector space," *Glas. mat.,* **13**, 169–189 (1958).
183. S. Kurepa, "A cosine functional equation in Banach algebras," *Acta. Sci. Math. Szeged.,* **23**, 255–267 (1962).
184. S. Kurepa, "A weakly measurable selfadjoint cosine functions," *Glas. mat.,* **8**, No. 1, 73–79 (1973).
185. S. Kurepa, "Decomposition of weakly measurable semigroups and cosine operatos functions," *Glas. mat.,* **3**, No. 11 (31), 91–95 (1976).
186. S. Kurepa, "Semigroups and cosine functions," *Funct. Anal. Lect. Notes Math.,* **948**, 47–72 (1982).
187. T. Kusano and S. Oharu, "Semilinear evolution equations with singularities in ordered Banach spaces," *Differ. Integral Equations*, **5**, No. 6, 1383–1405 (1992).
188. Ch. Le Merdy, "The similarity problem for bounded analytic semigroups on Hilbert space," *Semigroup Forum*, **56**, 205–224 (1998).
189. C. Le Merdy, "Counterexamples on L_p-maximal regularity," *Math. Z.*, **230**, 47–62, (1999).
190. M. Li, X. H. Gu, and F. L. Huang, "On unbounded perturbations of semigroups: compactness and norm continuity," *Semigroup Forum*, **65**, 58–70 (2002).

191. Miao Li, B.-Z. Guo, and S. Piskarev, "Compactness and norm continuity of the difference of two cosine functions," to appear in *Taiwan. J. Math.*

192. Y.-C. Li and S.-Y. Shaw, *Integrated C-Cosine Functions and the Abstract Cauchy Problem* (1991), preprint.

193. Liang Jin, Xiao Tijun, "Well-posedness results for certain classes of higher order abstract Cauchy problems connected with integrated semigroups," *Semigroup Forum,* **56**, 84–103 (1998).

194. Liang Jin, Xiao Tijun, "Norm continuity (for $t > 0$) of propagators of arbitary order abstract differential equations in Hilbert spaces," *J. Math. Anal. Appl.*, **204**, 124–137 (1996).

195. Zh. Liu and J. Yong, "Qualitative properties of certain C_0-semigroups arising in elastic systems with various damping," *Adv. Differ. Equat.*, **3**, No. 5, 643–686 (1998).

196. C. Lizama, "A characterization of periodic resolvent operators," *Res. Math.*, **18**, 93–105 (1990).

197. C. Lizama, "On positive strongly continuous cosine functions," *Math. Balcan., New. Ser.*, **4**, No. 1, 43–49 (1990).

198. C. Lizama, "Uniform continuity and compactness for resolvent families of operators," In: *Seminar Notes in Funct. Anal. and Partial Differ. Equations*, Baton Rouge, USA (1992–1993), pp. 29–43.

199. Gunter Lumer and Luc Paquet, "Semi-groupes holomorphes, produit tensoriel de semi-groupes et équations d'évolution," In: *Séminaire de Théorie du Potentiel, No. 4 (Paris, 1977/1978)*, pp. 156–177. Springer, Berlin, 1979.

200. G. Lumer and L. Paquet, "Semi-groupes holomorphes et équations d'évolution," *C. R. Acad. Sci. Paris Sér. A–B*, **284**, No. 4, 237–240 (1977).

201. A. Lunardi, "Analytic semigroups and optimal regularity in parabolic problems," In: *Progress in Nonlinear Differential Equations and their Applications*, Vol. 16, Birkhäuser Verlag, Basel (1995).

202. Z. H. Luo, B. Z. Guo, and O. Morgul, *Stability and Stabilization of Infinite Dimensional Systems with Applications,* Springer-Verlag, London–Berlin–Heidelberg (1999).

203. D. Lutz, "Compactness properties of operator cosine functions," *C. R. Math. Rept. Acad. Sci. Canada,* **2**, 277–280 (1980).

204. D. Lutz, "On bounded time-dependent perturbations of operator cosine functions," *Aequat. math.*, **23**, Nos. 2–3, 197–203 (1981).

205. D. Lutz, "Periodische operatorwertige Cosinusfunktionen," *Result. Math.,* **4**, No. 1, 75–83 (1981).

206. D. Lutz, "Some spectral properties of bounded operator cosine functions," *Math. Repts. Acad. Sci. Canada,* **4**, No. 2, 81–85 (1982).

207. D. Lutz, "Strongly continuous operator cosine functions," *Functional Analysis, Lecture Notes in Math.*, **948**, 73–97 (1982).

208. D. Lutz, "Über der Konvergenz operatorwertiger Cosinusfunktionen mit gestortenz infinitesimalen Erzeuger," *Period. Math. Hung.*, **14**, No. 1, 101–105 (1983).

209. D. Lutz, "Über operatorwertiger Lösungen der Funktionalgleichung der Cosinus," *Math. Z.*, **171**, 233–245 (1980).

210. D. Lutz, "Which operators generate cosine functions?" *Acad. Naz. Lincei. Rend. Cl. Fiz. Mat. Natur.,* **63**, No. 5, 314–317 (1977).

211. C. Martinez, M. Sanz, and L. Marco, "Fractional powers of operators," *J. Math. Soc. Japan,* **40**, No. 2, 331–347 (1988).

212. Alan McIntosh, "Square roots of elliptic operators," *J. Funct. Anal.*, **61**, No. 3, 307–327 (1985).

213. Alan McIntosh, "The square root problem for elliptic operators: a survey," In: *Functional-analytic methods for partial differential equations (Tokyo, 1989)*, pp. 122–140. Springer, Berlin (1990).

214. I. V. Mel'nikova, "Well-posedness of differential-operator problems. I. The Cauchy problem in spaces of distributions," *J. Math. Sci. (New York)*, **93**, No. 1, 1–21 (1999).

215. I. V. Mel'nikova, "Properties of the Lions d-semigroups, and generalized well-posedness of the Cauchy problem," *Funktsional. Anal. i Prilozhen.*, **31**, No. 3, 23–34, 95 (1997).

216. I. V. Mel'nikova and M. A. Al'shansky, "Well-posedness of Cauchy problem in Banach space: regular and degenerate cases," *J. Math. Science*, **87**, No. 4, 3732–3780 (1997).

217. I. V. Mel'nikova, U. A. Anufrieva, and V. Yu. Ushkov, "Degenerate distribution semigroups and well-posedness of the Cauchy problem," *Integral Transform. Spec. Funct.*, **6**, Nos. 1–4, 247–256 (1998).

218. P. D. Miletta, "Approximation of solutions to evolution equations," *Math. Methods Appl. Sci.,* **17**, 753–763 (1994).

219. T. Muramatu, "Besov spaces and analytic semigroups of linear operators," *J. Math. Soc. Japan,* **42**, No. 1, 133–146 (1990).

220. B. Nagy, "Approximation theorems for cosine operator functions," *Acta Math. Acad. Sci Hung.*, **29**, Nos. 1–2, 69–76 (1977).

221. B. Nagy, "Cosine operator functions and the abstract Cauchy problem," *Period. Math. Hung.,* **7**, Nos. 3–4, 213–217 (1976).

222. B. Nagy, "On cosine operator functions in Banach spaces," *Acta Sci. Math.*, **36**, Nos. 3–4, 281–289 (1974).

223. B. Nagy, "On the generation of cosine operator functions," *Publ. Math.*, **21**, Nos. 1–2, 151–154 (1974).

224. T. Naito and Nguyen Van Minh, "Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations," *J. Differ. Equat.,* **152**, 358–376 (1999).

225. S. Nelson and R. Triggiani, "Analytic properties of cosine operators," *Proc. Amer. Math. Soc.,* **74**, 101–104 (1979).

226. F. Neubrander, "On the relation between the semigroup and its infinitesimal generator," *Proc. Amer. Math. Soc.,* **100**, No. 1, 104–108 (1987).

227. Nguyen Thanh Lan, "On nonautonomous higher order Cauchy problems," *Preprint* (1998).

228. Nguyen Thanh Lan, "On the well-posedness of nonautonomous second order Cauchy problems," *Arbeitsberich Funktionalanalysis* (1998).

229. Y. Niechwiej, "Cosine functions of unbounded operators and second order differential equations," *RADOVI Matematicki.,* **6**, 195–204 (1990).

230. H. Oka, "A class of complete second order linear differential equations," *Proc. Amer. Math. Soc.*, **124**, No. 10, 3143–3150 (1996).

231. L. Paquet, "Equations d'evolution pour operateurs locaux et equations aus derivees partielles," *C. R. Acad. Sci., Paris,* **286**, No. 4, A215–A218 (1978).

232. C. Palencia and S. Piskarev, "On the multiplicative perturbations of C_0-groups and C_0-cosine operator functions," *Semigroup Forum,* **63**, No. 2, 127–152 (2001).

233. N. Pavel, "Invariant sets for a class of semilinear equations of evolution," *Nonlinear Anal. Theory Appl.*, **1**, No. 2, 187–196 (1977).

234. N. Pavel, "Second order differential equations on closed subspace of a Banach space," *Boll. Unione Mat. Ital.*, **12**, No. 4, 348–353 (1975).

235. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York (1983).

236. J. Peng, M. Wang, and G. Zhu, "The spectrum of the generator of a class of perturbed semigroups with applications" [in Chinese] *Acta Math. Appl. Sinica*, **20**, 107–113 (1997).

237. W. V. Petryshyn, *Approximation-Solvability of Nonlinear Functional and Differential Equations,* Marcel Dekker, New York–Basel–Hong Kong (1993).

238. S. Piskarev and S.-Y. Shaw, "Multiplicative perturbations of C_0-semigroups and some applications to step responses and cumulative outputs," *J. Funct. Anal.*, **128**, No. 2, 315–340 (1995).

239. S. Piskarev and S.-Y. Shaw, "On certain operator families related to cosine operator functions," *Taiwan. J. Math.*, **1**, 527–546 (1997).

240. S. Piskarev and S.-Y. Shaw, "Perturbation and comparison of cosine operator functions," *Semigroup Forum,* **51,** 225–246 (1995).

241. S. Piskarev and S.-Y. Shaw, “Perturbation of cosine operator functions by step response and cumulative output,” In: *Proc. Conf. Evolution Equations,* University of Strathclyde, July, 25–29, 1994.

242. N. Popescu, “Equations differentielles de deuxieme ordre et operateurs hermitien-equivalents dans les espaces de Banach,” In: *Proc. 4th Conf. Oper. Theory, Timisoara* (1979), pp. 12–22.

243. J. Prüss, “Evolutionary, integral equations and applications,” Birkhäuser, Basel (1993).

244. J. Prüss, “Positivity and regularity of hyperbolic Volterra equations in Banach spaces,” *Math. Ann.*, **279**, 317–344 (1987).

245. M. V. Radic and F. Vajzovic, “On the bounded cosine operator function in Banach space,” *Mat. Vesn.*, **39**, No. 2, 187–204 (1987).

246. S. M. Rankin, “A remark on cosine families,” *Proc. Amer. Math. Soc.*, **79**, 376–378 (1980).

247. B. P. Rao, “Uniform stabilization of a hybrid system of elasticity,” *SIAM J. Control Optim.*, **33**, 440–445 (1995).

248. R. Rabiger and W. Ricker, “C_0-semigroups and cosine families of linear operators in hereditarily indecomposable Banach spaces,” *Acta Sci. Math.,* **64**, No. 3–4, 697–706 (1998).

249. M. Reghis and Gh. Babescu, “Cosine and sine functions valued in Banach algebras (II): some spectral properties,” *Ann. Univ. Timisoara. Ser. Sti. Mat.-Fiz.*, **26**, No. 1, 93–96 (1988).

250. W. M. Ruess and Q. P. Vu, “Asymptotically almost periodic solutions of evolution equations in Banach spaces,” *J. Differ. Equat.*, **122**, 282–301 (1995).

251. D. L. Russell, “Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions,” *SIAM Review*, **20**, 639–739 (1978).

252. J. M. Rassias, “Six-dimensional Landau inequalities,” *Bull. Calcutta Math. Soc.,* **87**, No. 6, 473–494 (1997).

253. J. M. Rassias, “Multi-dimensional Landau extremum problems,” *J. R. Acad. Bulg. Sci.,* **50**, No. 2, 5–9 (1997).

254. J. T. Sandefur (Jr.), “Existence and uniqueness of solutions of second order nonlinear differential equations,” *SIAM J. Math. Anal.,* **14**, No. 3, 477–487 (1984).

255. H. Serisawa and M. Watanabe, “Perturbations for cosine families in Banach spaces,” *Houston J. Math.*, **12**, No. 1, 117–124 (1986).

256. H. Serisawa and M. Watanabe, “Time dependent perturbations for cosine families,” *Houston J. Math.*, **12**, No. 4, 579–586 (1986).

257. S.-Y. Shaw, “Mean and pointwise ergodic theorems for cosine operator functions,” *Math. J. Okajama Univ.*, **27**, 197–203 (1985).

258. S.-Y. Shaw, “Mean ergodic theorems and functional equations,” *J. Funct. Anal.*, **87**, 428–441 (1989).

259. S.-Y. Shaw, “On w^*-continuous cosine operator functions,” *J. Funct. Anal.*, **66**, No. 1, 73–95 (1986).

260. S.-Y. Shaw, C.-S. Lee, and W.-L. Chiou, “Representation formulas for cosine and sine functions of operators, II,” *Aequat. Math.,* **31**, No. 1, 64–75 (1986).

261. S.-Y. Shaw and S. Piskarev, “Asymptotic behaviour of semigroup-related functions,” *Saitama Math. J.,* **12**, 7–12 (1994).

262. M. Shimizu and I. Miyadera, “Perturbation theory for cosine families on Banach spaces,” *Tokyo J. Math.,* **1**, No. 2, 333–343 (1978).

263. A. Ya. Shklyar, “Complete second order linear differential equations in Hilbert spaces,” *Operator Theory Adv. and Appl.*, **92**, Birkhäuser (1997).

264. M. Sova, “Cosine operator functions,” *Rozpr. Mat.*, **49**, 1–47 (1966).

265. M. Sova, “Linear differential equations in Banach spaces,” *Rozpr. CSAV MPV*, **85**, No. 6, 1–86 (1975).

266. M. Sova, “On exponential growth of solutions of abstract initial value problems,” *Cas. Pestov. Mat.*, **106**, No. 1, 1–30 (1981).

267. H. B. Stewart, “Generation of analytic semigroups by strongly elliptic operators,” *Trans. Amer. Math. Soc.,* **199**, 141–162 (1974).

268. H. B. Stewart, “Generation of analytic semigroups by strongly elliptic operators under general boundary conditions,” *Trans. Amer. Math. Soc.*, **259**, 299–310 (1980).

269. T. Takenaka, “A note on the generation theorem of cosine families in Banach spaces,” *Met. Coll. Eng., Chubu Univ.*, **22**, 71–76 (1986).

270. T. Takenaka and N. Okazawa, “A Phillips–Miyadera type perturbation theorem for cosine functions of operators,” *Tohoku Math. J.*, **30**, 107–115 (1978).

271. C. C. Travis, “Differentiability of weak solutions to an abstract inhomogenious differential equation,” *Proc. Amer. Math. Soc.*, **82**, 425–430 (1981).

272. C. C. Travis and G. F. Webb, “Second order differential equations in Banach space,” *Nonlin. Equat. in Abstract Space,* 331–361 (1978), *Proc. Int. Symp.,* Arlington, 1977.

273. C. C. Travis and G. F. Webb, “Compactness, regularity and uniform continuity properties of strongly continuous cosine families,” *Houston J. Math.*, **3**, No. 4, 555–567 (1977).

274. C. C. Travis and G. F. Webb, “Cosine families and abstract non-linear second order differential equations,” *Acta Math. Acad. Sci. Hung.,* **32**, Nos. 3–4, 75–96 (1978).

275. C. C. Travis and G. F. Webb, “Perturbation of strongly continuous cosine family generators,” *Colloq. Math.*, **45**, No. 2, 277–285 (1981).

276. R. Triggiani, “Lack of uniform stabilization for noncontractive semigroups under compact perturbation,” *Proc. Amer. Math. Soc.,* **105**, 375–383 (1989).

277. S. Tymovski, “On the analogue of the formula $\cos(t) = (e^t + e^{-t})/2$ for operator cosine function,” *Rocz. Ptm.*, Ser. 1, **23**, No. 1, 173–182 (1983).

278. G. Vainikko, “Approximative methods for nonlinear equations (two approaches to the convergence problem,” *Nonlinear Anal., Theory, Meth. and Appl.*, **2**, 647–687 (1978).

279. F. Vajzovic, “On the cosine operator function,” *Glas. mat.*, **22**, No. 2, 381–406 (1987).

280. F. Vajzovic, “On cosine operator functions in Banach spaces,” *Rad. Acad. Nauka i Umjetn. bih od. prirod. i mat. Nauka,* **62**, No. 20, 23–30 (1982).

281. J. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators,* Birkhäuser Verlag (1996).

282. J. van Neerven, “Hahn–Banach type theorem for adjoint semigroups,” *Math. Ann.,* **28**, 63–71 (1990).

283. I. Vidav, “Spectral of perturbed semigroups with applications to transport theory,” *J. Math. Anal. Appl.*, **30**, 264–279 (1970).

284. J. Voigt, “On convex compactness property for strong operator topology,” *Notes di Matematika,* **12**, 259–269 (1992).

285. J. Warga, *Optimal Control of Differential Equations,* Academic Press, New York (1972).

286. M. Watanabe, “A new proof of the generation theorem of cosine families in Banach spaces,” *Houston J. Math.,* **10**, No. 2, 285–290 (1984).

287. M. Watanabe, “A perturbation theory for abstract evolution equations of second order,” *Proc. Jap. Acad.*, **58**, 143–146 (1982).

288. M. Watanabe, “Cosine families of operators and applications,” *Differ. Equat. in Banach spaces: Lect. Notes in Math.*, **1223**, 278–292 (1986).

289. G. F. Webb, “A representation formula for strongly continuous cosine families,” *Aequat. Math.*, **20**, No. 1, 115–116 (1980).

290. G. F. Webb, “Compactness of bounded trajectories of dynamic systems in infinite dimensional systems,” *Proc. Roy. Soc., Edinburgh,* **84A**, 19–33 (1979).

291. G. F. Webb, “An operator-theoretic formulation of asynchronic exponential growth,” *Trans. Amer. Math. Soc.*, **303**, 751–763 (1987).

292. L. Weis, “A new approach to maximal L_p regularity,” In: *Proc. 6th Internat. Conf. on Evolution Equations*, Marcel Dekker (2000).

293. L. Weis, “Operator-valued multiplier theorems and maximal L_p regularity,” *Mathematische Annalen*, **319**, 735–758, (2001).

294. T. Xiao and J. Liang, "Analyticity of the propagators of second order linear differential equations in Banach spaces," *Semigroup Forum,* **44**, 356–363 (1992).
295. T. Xiao and J. Liang, "Entire solutions of higher order abstract Cauchy problems," *J. Math. Anal. and Appl.,* **208**, 298–310 (1997).
296. T. Xiao and J. Liang, "Differential operators and C-well-posedness of complete second order abstract Cauchy problems," *Pacif. J. Math.*, **186**, No. 1, 167–200 (1998).
297. T. Xiao and J. Liang, "The Cauchy problem for higher order abstract differential equations," *Lect. Notes in Math.,* **1701** (1998).
298. P. H. You, "Characteristic conditions for a C_0-semigroup with continuity in the uniform operator topology for $t > 0$ in Hilbert space," *Proc. Amer. Math. Soc.*, **116**, 991–997 (1992).
299. S. Zaidman, *Topics in Abstract Differential Equations,* Pitman, Vol. 304 (1994).
300. S. Zaidman, *Topics in Abstract Differential Equations II,* Pitman, Vol. 321 (1994).
301. S. Zaidman, *Functional Analysis and Differential Equations in Abstract Spaces,* Chapman & Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics, Vol. 100 (1999).
302. Q. C. Zhang and F. L. Huang, "On compact perturbations of C_0-semigroups" [in Chinese], *J. Sichuan Univ.* **35**, 829–833 (1998).